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A FINITE DIFFERENCE SCHEME FOR COMBINED KdV-MKdV EQUATION

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Abstract: In this paper, a finite difference method for a Cauchy problem of combined KdV-MKdV equation was considered. An energy conservative finite difference scheme was proposed. Convergence and stability of the difference solution were proved. Therefore, this method is efficient and reliable.

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1. Introduction

The KdV and MKdV equations are most popular soliton equations and have been extensively investigated. But the nonlinear terms of KdV and MKdV equations often simultaneously exist in practical problems such as fluid physics, physics and quantum field theory, and form the following so-called combined KdV-MKdV equation

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} = 0, \tag{1}$$

where α and β are arbitrary constants. This equation may describe the wave propagation of the bound particle, sound wave and thermal pulse [1], [2]. Recently, The numerical solutions of equation (1) in [3] have been given.

$$u_t + u_x + \alpha(u^p)_x - \beta u_{xxt} = 0, \tag{2}$$

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Which is usually called the generalized regularized long-wave (GRLW) equation, has been studied, too. Doğan Kaya [4] gives the exact solitary-solutions of GRLW equation (3), and construct numerical solutions using ADM (Adomian decomposition method) without using any discretization technique. Luming Zhang gives a finite difference scheme for generalized regularized long-wave equation in [6]. Motivated by Zhang's work, we consider the following initial value problem of combined KdV-MKdV equation,

$$u_t + \alpha u u_x + \beta u^2 u_x + \gamma u_{xxx} = 0, \tag{3}$$

$$u|_{t=0} = u_0(x). \tag{4}$$

2. Finite Difference Scheme and Conservation Law

As usual, the following notations will be used

$$(u_j^n)_x = \frac{u_{j+1}^n - u_j^n}{h}, \quad (u_j^n)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h},$$

$$(u_j^n)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \quad (u_j^n)_{\hat{t}} = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau},$$

$$(u^n, v^n) = h \sum_j u_j^n v_j^n, \quad \|u^n\|^2 = (u^n, u^n), \quad \|u^n\|_\infty = \sup_j |u_j^n|,$$

where h and τ are the spatial and temporal step sizes, respectively, and $x_j = jh, t_n = n\tau$. Superscript n denotes a quantity associated with the time-level t_n and Superscript j denotes a quantity associated with space mesh point x_j. In this paper, C denotes general constant, which may have different value in different place.

Now we consider the finite difference method for the problem (3), (4). Since

$$(u^2)_x = \frac{2}{3}[u u_x + (u^2)_x],$$

then the following finite difference scheme is considered.

$$(u_j^n)_{\hat{t}} + \frac{\alpha}{6}[u_j^n[(u_j^{n+1})_{\hat{x}} + (u_j^{n-1})_{\hat{x}}] + [u_j^n(u_j^{n+1} + u_j^{n-1})]_{\hat{x}}] + \frac{\beta}{8}[(u_j^n)^2[(u_j^{n+1})_{\hat{x}} + (u_j^{n-1})_{\hat{x}}] + [(u_j^n)^2(u_j^{n+1} + u_j^{n-1})]_{\hat{x}}] + \frac{\gamma}{2}[(u_j^{n+1})_{x\bar{x}\hat{x}} + (u_j^{n-1})_{x\bar{x}\hat{x}}] = 0, \tag{5}$$

$$u_j^0 = u_0(jh). \tag{6}$$

Lemma 1. *The finite difference scheme (5), (6) is conservative for discrete* *energy, i.e.,*

$$E^n = \|u^{n+1}\|^2 + \|u^n\|^2 = E^{n-1} = ... = E_0. \qquad (7)$$

Proof. Computing the inner product of (5) with $u^{n+1} + u^{n-1}$, we obtain

$$h\sum_j (u_j^n)_{\hat{t}}(u_j^{n+1} + u_j^{n-1}) + \frac{\alpha}{6}h\sum_j [u_j^n((u_j^{n+1})_{\hat{x}} + (u_j^{n-1})_{\hat{x}}) + [u_j^n(u_j^{n+1} + u_j^{n-1})]_{\hat{x}}](u_j^{n+1} + u_j^{n-1}) + \frac{\beta}{8}h\sum_j [(u_j^n)^2[(u_j^{n+1})_{\hat{x}} + (u_j^{n-1})_{\hat{x}}] + [(u_j^n)^2(u_j^{n+1} + u_j^{n-1})]_{\hat{x}}](u_j^{n+1} + u_j^{n-1}) + \frac{\gamma}{2}h\sum_j [(u_j^{n+1})_{x\bar{x}\hat{x}} + (u_j^{n-1})_{x\bar{x}\hat{x}}](u_j^{n+1} + u_j^{n-1}) = 0, \qquad (8)$$

where

$$I = h\sum_j (u_j^n)_{\hat{t}}(u_j^{n+1} + u_j^{n-1}) = \frac{1}{2\tau}(\|u^{n+1}\|^2 - \|u^{n-1}\|^2), \qquad (9)$$

$$\begin{aligned}
II &= \frac{\alpha}{6}h\sum_j [u_j^n(u_j^{n+1} + u_j^{n-1})_{\hat{x}} + [u_j^n(u_j^{n+1} + u_j^{n-1})]_{\hat{x}}](u_j^{n+1} + u_j^{n-1}) \\
&= \frac{\alpha}{12}\sum_j [u_j^n(u_{j+1}^{n+1} - u_{j-1}^{n+1} + u_{j+1}^{n-1} - u_{j-1}^{n-1}) \\
&\quad + u_{j+1}^n(u_{j+1}^{n+1} + u_{j+1}^{n-1}) - u_{j-1}^n(u_{j-1}^{n+1} + u_{j-1}^{n-1})](u_j^{n+1} + u_j^{n-1}) \\
&= \frac{\alpha}{12}\sum_j [u_j^n(u_{j+1}^{n+1} + u_{j+1}^{n-1}) + u_{j+1}^n(u_{j+1}^{n+1} + u_{j+1}^{n-1})](u_j^{n+1} + u_j^{n-1}) \\
&- \frac{\alpha}{12}\sum_j [u_j^n(u_{j-1}^{n+1} + u_{j-1}^{n-1}) + u_{j-1}^n(u_{j-1}^{n+1} + u_{j-1}^{n-1})](u_j^{n+1} + u_j^{n-1}) \\
&= \frac{\alpha}{12}\sum_j [u_j^n(u_{j+1}^{n+1} + u_{j+1}^{n-1}) + u_{j+1}^n(u_{j+1}^{n+1} + u_{j+1}^{n-1})](u_j^{n+1} + u_j^{n-1}) \\
&- \frac{\alpha}{12}\sum_j [u_{j+1}^n(u_j^{n+1} + u_j^{n-1}) + u_j^n(u_j^{n+1} + u_j^{n-1})](u_{j+1}^{n+1} + u_{j+1}^{n-1}) \\
&= 0, \qquad (10)
\end{aligned}$$

$$\begin{aligned}
III &= \frac{\beta}{8}h\sum_j [(u_j^n)^2[(u_j^{n+1})_{\hat{x}} + (u_j^{n-1})_{\hat{x}}] + [(u_j^n)^2(u_j^{n+1} + u_j^{n-1})]_{\hat{x}}](u_j^{n+1} + u_j^{n-1}) \\
&= \frac{\beta}{16}\sum_j [(u_j^n)^2(u_{j+1}^{n+1} + u_{j+1}^{n-1} - u_{j-1}^{n+1} - u_{j-1}^{n-1})
\end{aligned}$$

$$
\begin{aligned}
&+(u_{j+1}^n)^2(u_{j+1}^{n+1}+u_{j+1}^{n-1})-(u_{j-1}^n)^2(u_{j-1}^{n+1}+u_{j-1}^{n-1})](u_j^{n+1}+u_j^{n-1})\\
&=\frac{\beta}{16}\sum_j[(u_j^n)^2(u_{j+1}^{n+1}+u_{j+1}^{n-1})+(u_{j+1}^n)^2(u_{j+1}^{n+1}+u_{j+1}^{n-1})](u_j^{n+1}+u_j^{n-1})\\
&-\frac{\beta}{16}\sum_j[(u_{j+1}^n)^2(u_j^{n+1}+u_j^{n-1})+(u_j^n)^2(u_j^{n+1}+u_j^{n-1})](u_{j+1}^{n+1}+u_{j+1}^{n-1})\\
&=0, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
IV&=\frac{\gamma}{2}h\sum_j[(u_j^{n+1})_{x\bar{x}\hat{x}}+(u_j^{n-1})_{x\bar{x}\hat{x}}](u_j^{n+1}+u_j^{n-1})\\
&=\frac{\gamma}{2h}\sum_j[(u_{j+1}^{n+1})_{\hat{x}}-2(u_j^{n+1})_{\hat{x}}+(u_{j-1}^{n+1})_{\hat{x}}\\
&\quad+(u_{j+1}^{n-1})_{\hat{x}}-2(u_j^{n-1})_{\hat{x}}+(u_{j-1}^{n-1})_{\hat{x}}](u_j^{n+1}+u_j^{n-1})\\
&=\frac{\gamma}{4h^2}\sum_j[u_{j+2}^{n+1}-u_j^{n+1}-2u_{j+1}^{n+1}+2u_{j-1}^{n+1}+u_j^{n+1}-u_{j-2}^{n+1}\\
&\quad+u_{j+2}^{n-1}-u_j^{n-1}-2u_{j+1}^{n-1}+2u_{j-1}^{n-1}+u_j^{n-1}-u_{j-2}^{n-1}](u_j^{n+1}+u_j^{n-1})\\
&=\frac{\gamma}{4h^2}\sum_j[u_{j+2}^{n+1}u_j^{n+1}-2u_{j+1}^{n+1}u_j^{n+1}+2u_{j-1}^{n+1}u_j^{n+1}-u_{j-2}^{n+1}u_j^{n+1}\\
&\quad+u_{j+2}^{n-1}u_j^{n-1}-2u_{j+1}^{n-1}u_j^{n-1}+2u_{j-1}^{n-1}u_j^{n-1}-u_{j-2}^{n-1}u_j^{n-1}]\\
&=\frac{\gamma}{4h^2}\sum_j[u_{j+2}^{n+1}u_j^{n+1}-2u_{j+1}^{n+1}u_j^{n+1}+2u_j^{n+1}u_{j+1}^{n+1}-u_j^{n+1}u_{j+2}^{n+1}\\
&\quad+u_{j+2}^{n-1}u_j^{n-1}-2u_{j+1}^{n-1}u_j^{n-1}+2u_j^{n-1}u_{j+1}^{n-1}-u_j^{n-1}u_{j+2}^{n-1}]=0\,. \qquad (12)
\end{aligned}
$$

Substiting (9), (10), (11) and (12) into (8), and letting

$$
E^n=\|u^{n+1}\|^2+\|u^n\|^2, \qquad (13)
$$

we obtain (7).

3. Convergence and Stability of Finite Difference Scheme

First, we consider the truncation error of the finite difference scheme (5), (6). Suppose $v_j^n=u(x_j,t_n)$, then we have

$$
\begin{aligned}
Er_j^n&=(v_j^n)_{\hat{t}}+\frac{\gamma}{2}[(v_j^{n+1})_{x\bar{x}\hat{x}}+(v_j^{n-1})_{x\bar{x}\hat{x}}]+\frac{\alpha}{6}[v_j^n(v_j^{n+1}+v_j^{n-1})_{\hat{x}}\\
&\quad+[v_j^n(v_j^{n+1}+v_j^{n-1})]_{\hat{x}}]+\frac{\beta}{8}[(v_j^n)^2(v_j^{n+1}+v_j^{n-1})_{\hat{x}}
\end{aligned}
$$

$$+ [(v_j^n)^2(v_j^{n+1} + v_j^{n-1})]_{\hat{x}}] = 0, \quad (14)$$

According to Taylor's expansion, it can be easily obtained that linear part of (14) at point (x_j, t_n) satisfies

$$Er_j^n = O(h^2 + \tau^2).$$

Lemma 2. *Assume $u(x,t)$ is enough smooth, then the local truncation error of finite difference scheme (5), (6) is $O(h^2 + \tau^2)$.*

Next, we consider convergence and stability of the finite scheme (5), (6).

Lemma 3. (Discrete Sobolev's Inequality [5]) *For any discrete function u_h and for any given $\varepsilon > 0$, there exists a constant $K(\varepsilon, n)$, depending only ε and n, such that*

$$\|u^n\|_\infty \leq \epsilon\|u_x^n\| + K(\varepsilon, n)\|u^n\|.$$

Lemma 4. (see [5]) *Suppose that the discrete function w_h satisfies recurrence formula*

$$w_n - w_{n-1} \leq A\tau w_n + B\tau w_{n-1} + c_n\tau,$$

where A, B and c_n $(n = 1, ..., N)$ are nonnegative constants. Then

$$\|w_h\|_\infty \leq (w_0 + \tau\sum_{k=1}^{N} c_k)e^{2(A+B)\tau},$$

where τ is small, such that $(A+B)\tau \leq \frac{N-1}{2N}(N > 1)$.

Lemma 5. *Assume $u_0 \in H_0^1, |u_x x^2| < c, u_x$ is a monotony increasing function in $(-\infty, 0)$ and it is an even function in $(-\infty, +\infty)$, then there is the estimation for the solution of the difference scheme (5), (6),*

$$\|u^n\| \leq c, \|u_x^n\| \leq c, \|u^n\|_\infty \leq c.$$

Proof. We obtain from (7) $\|u^n\| \leq c$. According to $|u_x x^2| < c$, $(u_x^n, u_x^n) = \|u_x^n\|^2$, we get

$$\int_{-\infty}^{+\infty} |u_x|^2 dx = \int_1^{+\infty} |u_x|^2 dx + \int_{-\infty}^{-1} |u_x|^2 dx + \int_{-1}^{1} |u_x|^2 dx$$
$$\leq \int_1^{+\infty} \frac{c}{x^4} dx + \int_{-\infty}^{-1} \frac{c}{x^4} dx + \int_{-1}^{1} \frac{c}{x^4} dx \leq c\,. \quad (15)$$

Since u_x is a monotony increasing function in $(-\infty, 0)$ and it is an even function in $(-\infty, +\infty)$, we have

$$h\sum_j |(u_j^n)_x|^2 = 2h\sum_{j=0}^{+\infty} |(u_j^n)_x|^2 = (u_0^n)_x h + 2h\sum_{j=1}^{+\infty} |(u_j^n)_x|^2$$
$$\le c_1 + \sum_{j=1}^{+\infty}\int_{x_{j-1}}^{x_j} |u_x|^2 dx = c_1 + 2\int_0^{+\infty} |u_x|^2 dx \le c \quad (16)$$

Therefore,

$$\|u_x^n\| \le c$$

It follows from Lemma 4 that

$$\|u^n\|_\infty \le c.$$

Theorem 1. *Assume $u_0(x) \in H_0^1, u \in c^{(5,3)}, |u_x x^2| < c, u_x$ is a monotony increasing function in $(-\infty, 0)$ and it is an even function in $(-\infty, +\infty)$, then the solution of the conservative difference scheme (5), (6) converges to the solution of the problem (3), (4) with order $O(h^2+\tau^2)$ by l_∞ norm.*

Proof. Let $e_j^n = v_j^n - u_j^n$. Subtracting (5) from (14), we have

$$Er_j^n = (e_j^n)_{\hat{t}} + \frac{\gamma}{2}[(e_j^{n+1})_{x\bar{x}\hat{x}} + (e_j^{n-1})_{x\bar{x}\hat{x}}] + \frac{\alpha}{6}[v_j^n(v_j^{n+1}+v_j^{n-1})_{\hat{x}}$$
$$+ [v_j^n(v_j^{n+1}+v_j^{n-1})]_{\hat{x}} - u_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}}$$
$$- [u_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}] + \frac{\beta}{8}[(v_j^n)^2(v_j^{n+1}+v_j^{n-1})_{\hat{x}}$$
$$+ [(v_j^n)^2(v_j^{n+1}+v_j^{n-1})]_{\hat{x}} - (u_j^n)^2(u_j^{n+1}+u_j^{n-1})_{\hat{x}} - [(u_j^n)^2(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}]\,. \quad (17)$$

Computing the inner product of above equation with $e^{n+1}+e^{n-1}$, we get

$$(Er_j^n, e^{n+1}+e^{n-1})$$
$$= h\sum_j (e_j^n)_{\hat{t}}(e_j^{n+1}+e_j^{n-1}) + \frac{\gamma}{2}h\sum_j[(e_j^{n+1})_{x\bar{x}\hat{x}} + (e_j^{n-1})_{x\bar{x}\hat{x}}](e_j^{n+1}+e_j^{n-1})$$
$$+ \frac{\alpha}{6}h\sum_j[v_j^n(v_j^{n+1}+v_j^{n-1})_{\hat{x}} + [v_j^n(v_j^{n+1}+v_j^{n-1})]_{\hat{x}} - u_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}}$$
$$- [u_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}](e_j^{n+1}+e_j^{n-1}) + \frac{\beta}{8}h\sum_j[(v_j^n)^2(v_j^{n+1}+v_j^{n-1})_{\hat{x}}$$
$$+ [(v_j^n)^2(v_j^{n+1}+v_j^{n-1})]_{\hat{x}} - (u_j^n)^2(u_j^{n+1}+u_j^{n-1})_{\hat{x}}$$
$$- [(u_j^n)^2(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}](e_j^{n+1}+e_j^{n-1}), \quad (18)$$

$$I = \frac{1}{2\tau}(\|e^{n+1}\|^2 - \|e^{n-1}\|^2), \quad II = 0\,, \quad (19)$$

$$\begin{aligned}
III &= \frac{h\alpha}{6}\sum_j [v_j^n(v_j^{n+1}+v_j^{n-1})_{\hat{x}} + [v_j^n(v_j^{n+1}+v_j^{n-1})]_{\hat{x}} \\
&\quad - u_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}} - [u_j^n(u_j^{n+1}+u_j^{n-1})]_{\hat{x}}](e_j^{n+1}+e_j^{n-1}) \\
&= \frac{h\alpha}{6}\sum_j [v_j^n(e_j^{n+1}+e_j^{n-1})_{\hat{x}} + e_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}}] \\
&\quad + \frac{\alpha}{12}\sum_j [v_{j+1}^n(v_{j+1}^{n+1}+v_{j+1}^{n-1}) - v_{j-1}^n(v_{j-1}^{n+1}+v_{j-1}^{n-1}) - u_{j+1}^n(u_{j+1}^{n+1}+u_{j+1}^{n-1}) \\
&\quad + u_{j-1}^n(u_{j-1}^{n+1}+u_{j-1}^{n-1})](e_j^{n+1}+e_j^{n-1}) \\
&= \frac{h\alpha}{6}\sum_j [v_j^n(e_j^{n+1}+e_j^{n-1})_{\hat{x}} + e_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}}](e_j^{n+1}+e_j^{n-1}) \\
&\quad + \frac{\alpha}{12}\sum_j [v_{j+1}^n(e_{j+1}^{n+1}+e_{j+1}^{n-1}) + e_{j+1}^n(u_{j+1}^{n+1}+u_{j+1}^{n-1}) \\
&\quad - v_{j-1}^n(e_{j-1}^{n+1}+e_{j-1}^{n-1}) - e_{j-1}^n(u_{j-1}^{n+1}+u_{j-1}^{n-1})](e_j^{n+1}+e_j^{n-1}) \\
&= \frac{h\alpha}{6}\sum_j [v_j^n(e_j^{n+1}+e_j^{n-1})_{\hat{x}} + e_j^n(u_j^{n+1}+u_j^{n-1})_{\hat{x}} \\
&\quad + [v_j^n(e_j^{n+1}+e_j^{n-1})]_{\hat{x}} + [e_j^n(u_j^{n+1}+u_j^{n+1})]_{\hat{x}}](e_j^{n+1}+e_j^{n-1}) \\
&\le c[\|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2 + \|e_x^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2]. \qquad (20)
\end{aligned}$$

Similarly,

$$IV \le c[\|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2 + \|e_x^n\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2].$$

In addition, there exist obviously that

$$(Er^n, e^{n+1}+e^{n-1}) \le \|Er^n\|^2 + \tfrac{1}{2}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2). \qquad (21)$$

Substituting (19)-(21) into (18), we get

$$\begin{aligned}
\frac{1}{2\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) &\le \|Er^n\|^2 + \frac{1}{2}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2) \\
&\quad + c(\|e_x^{n+1}\|^2 + \|e_x^n\|^2 + \|e_x^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2). \qquad (22)
\end{aligned}$$

$$\|u_x^n\| \le c, \qquad \|e_x^n\| \le c,$$

so,

$$\begin{aligned}
&\frac{1}{2\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) \\
&\le \|Er^n\|^2 + \frac{1}{2}(\|e^{n+1}\|^2 + \|e^{n-1}\|^2) + c(\|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2). \qquad (23)
\end{aligned}$$

Let

$$B^n = \frac{1}{2}(\|e^{n+1}\|^2 + \|e^n\|^2)\,,$$

then (23) can be rewritten as

$$B^n - B^{n-1} \leq \tau\|Er^n\|^2 + c\tau(B^n + B^{n-1}). \tag{24}$$

By Lemma 4 it can immediately be obtained that

$$B^N \leq (B^0 + T \sup_{1\leq n\leq N} \|Er^n\|^2)e^{cT}. \tag{25}$$

Thus we can choose a two order method to compute u^1 such that

$$B^0 \leq [O(h^2 + \tau^2)]^2$$

It follows from (25) that

$$\|e^n\| \leq O(h^2 + \tau^2), \quad \|e^n_x\| \leq O(h^2 + \tau^2).$$

According to Lemma 4 there exist that

$$\|e^n\|_\infty \leq O(h^2 + \tau^2). \qquad \square$$

Similarly, it can be proved that

Theorem 2. *Under the conditions of Theorem 1, the solution of conservative finite difference scheme (5), (6) is stable by l_∞ norm.*

4. Conclusions

We have illustrated how an energy conservative finite difference scheme can be used to solve combined KdV-MKdV equation. Convergence and stability of the difference solution were proved. According to the illustration, we can draw a conclusion that this method is efficient and reliable.

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