International Journal of Pure and Applied Mathematics
Volume 41 No. 5 2007, 713-734

GROWTH OF POLYNOMIALS NOT VANISHING IN A DISK OF PRESCRIBED RADIUS

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Abstract: In this paper, we consider the class of polynomials $P(z) = a_0 + a_\mu z^\mu + \cdots + a_n z^n$, $1 \leq \mu \leq n$, of degree n not vanishing in the disk $|z| < k$. For $k \geq 1$, we investigate the dependance of $\text{Max}_{|z|=1}|P(Rz) - P(rz)|$, $R \geq r \geq 1$, on $\text{Max}_{|z|=1}|P(z)|$ and $\text{Min}_{|z|=k}|P(z)|$. For any given complex number β and $k > 0$, we also measure the growth of $\text{Max}_{|z|=1}|P(Rz) - \beta P(rz)|$ and the growth of $\text{Max}_{|z|=R}|P(\rho z) - P(z)|$, $\rho > 1$, where $0 \leq r \leq R \leq k$. Our results contitute multifaceted generalizations which besides yielding several interesting results as corollaries also lead to some striking conclusions giving extensions and refinements of some known polynomial inequalities.

AMS Subject Classification: 30A10, 30C10, 30C15
Key Words: polynomials, inequalities, maximum modulus, growth

1. Introduction and Statments of Results

If P(z) is a polynomial of degree n, then concerning the estimate of the maximum of $|P'(z)|$ on the unit circle $|z| = 1$ and the estimate of the maximum of $|P(z)|$ on a larger circle $|z| = R > 1$, we have

$$\text{Max}_{|z|=1}|P'(z)| \leq \text{Max}_{|z|=1}|P(z)| \tag{1}$$

and

$$\text{Max}_{|z|>1}|P(z)| \leq R^n \text{Max}_{|z|=1}|P(z)|\,. \tag{2}$$

Received: August 9, 2007

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The first inequality is better known as S. Bernstein's inequality (for refernce see [22], [15] or [19]), although it first appeared in a paper of M. Riesz [20, p. 357]. The second inequality is a simple deduction from the (2) equality holds only for $P(z) = \alpha z^n, \alpha \neq 0$.

If we apply the inequality (2) to the polynomial $P(rz)$ where $0 < r < 1$, we get

$$\mathrm{Max}_{|z|=1}|P(Rrz)| = \mathrm{Max}_{|z|>1}|P(rz)| \leq R^n \mathrm{Max}_{|z|=1}|P(rz)|.$$

Taking $R = \frac{1}{r}$, then $R > 1$ and it follows that

$$\mathrm{Max}_{|z|=r<1}|P(z)| \geq r^n \mathrm{Max}_{|z|=1}|P(z)|. \tag{3}$$

Varga [24, p. 44] attributes inequality (3) to E.H. Zarantonell.

If $P(z) \neq 0$ for $|z| < 1$, then inequalities (1), (2) and (3) can be replaced by

$$\mathrm{Max}_{|z|=1}|P(z)| \leq \frac{n}{2}\mathrm{Max}_{|z|=1}|P(z)| - \mathrm{Min}_{|z|=1}|P(z)|\,, \tag{4}$$

$$\mathrm{Max}_{|z|=1}|P(z)| \leq \frac{R^n+1}{2}\mathrm{Max}_{|z|=1}|P(z)| \tag{5}$$

and

$$\mathrm{Max}_{|z|=1}|P(z)| \leq \left(\frac{1+r}{2}\right)^n \mathrm{Max}_{|z|=1}|P(z)| - \mathrm{Min}_{|z|=1}|P(z)|. \tag{6}$$

Inequality (4) was conjectured by P. Erdös and later verified by P.D. Lax [13], where as Ankeny and Rivlin[1] used (4) to prove the inequality, see [5]. The bound in (4) is attained for those polynomials of degree n which have all their zeros on $|z| = 1$ and inequality (5) becomes an inequality for polynomials of the form $P(z) = \alpha(z^n + \beta), \alpha \neq 0, |\beta| = 1$. Inequality (6), which was proved by Rivlin [21], is also sharp with equality for polynomials of the form $P(z) = \alpha(z+\beta)^n, \alpha \neq 0, |\beta| = 1$.

Inequalities (4) and (5) were further improved by Aziz and Dawood (see [3], also [27]). In fact, if $P(z) \neq 0$ for $|z| < 1$, they proved that

$$\mathrm{Max}_{|z|=1}|P^{'}(z)| \leq \frac{n}{2}\left(\mathrm{Max}_{|z|=1}|P(z)| - \mathrm{Min}_{|z|=1}|P(z)|\right) \tag{7}$$

and

$$\mathrm{Max}_{|z|=1}|P(z)| \leq \frac{R^n+1}{2}\mathrm{Max}_{|z|=1}|P(z)| - \left(\frac{R^n-1}{2}\right) - \mathrm{Min}_{|z|=1}|P(z)|. \tag{8}$$

Extensions of the inequalities (4), (5) and (6) were obtained among others by Malik in [14], Govil, Rehman and Schmeisser in [12], Govil in [10] and Qazi in [18]. As a generalization of (4) it was shown by Chan and Malik [7] that if $P(z) = a_0 + \sum_{j=p}^{n} a_j z^j, 1 \leq \mu \leq n$, is a polynomial of degree n which deos not vanish in the disk $|z| < k$, where $k \geq 1$, then

$$\text{Max}_{|z|=1}|P^{'}(z)| \leq \frac{n}{1+k^n}\text{Max}_{|z|=1}|P(z)|. \tag{9}$$

Inequality (9) was independently proved by Qazi [18, Lemma 1], who under the same hypothesis has shown that

$$\text{Max}_{|z|=1}|P^{'}(z)| \leq \frac{n}{1+k^n\phi(\mu,k)}\text{Max}_{|z|=1}|P(z)|\,, \tag{10}$$

where

$$\phi(\mu,k) = \frac{k+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^\mu}{1+\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^{\mu+1}} \tag{11}$$

and

$$\frac{\mu}{n}|\frac{a_\mu}{a_0}|k^n \leq 1, \quad 1 \leq \mu \leq n.$$

Clearly $\phi(\mu,k) \geq 1$ for $k \geq 1$ and $1 \leq \mu \leq n$. Hence (10) is a refinement of (9). For $\mu = 1$, inequality (9) is due to Malik [14] and inequality (10) was proved by Govil, Rahman and Schmeisser [12]. As a generalization of (6), Qazi [18, Theorem 1] used inequality (9) to prove that if $P(z) = a_0 + \sum\limits_{j=p}^{n} a_j z^j, 1 \leq \mu \leq n$, has no zeros in $|z| < 1$, then for $0 \leq r \leq 1$,

$$\text{Max}_{|z|=R}|P(z)| \leq \left(\frac{1+R^\mu}{1+r^n}\right)^{\frac{n}{\mu}}\text{Max}_{|z|=r}|P(z)|. \tag{12}$$

For $R = 1 = \mu$, (12) reduces to the inequality (6).

Recently, Frappier, Rahman and Ruscheweyh [19, Theorem 1] investigated the dependence of

$$\text{Max}_{|z|=1}|P(Rz) - P(z)| \quad \text{on} \quad \text{Max}_{|z|=1}|P(z)|\,,$$

where $R > 1$ and proved that if P(z) is a plynomial of degree n, then

$$\text{Max}_{|z|=1}|P(Rz) - P(z)| + \psi_n(R)|P(0)| \leq (R^n - 1)\text{Max}_{|z|=1|}P(z)|, \tag{13}$$

where

$$\psi_n(R) = \frac{(R-1)(R^{n-1}+R^{n-2})\left(R^{n+1}+R^n-(n+1)R+n-1\right)}{R^{n+1}+R^n-(n-1)R+(n-3)},\ n \geq 2$$

and

$$\psi_1(R) = R - 1.$$

In this paper, we consider for a fixed μ, the class of polynomials

$$\Re_{n,\mu} := \left(P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \quad 1 \leq \mu \leq n\right)$$

not vanishing in the disk $|z| < k$, where $k \geq k$, and investigate the dependence of

$$\text{Max}_{|z|=1}|P(Rz) - P(z)| \text{ on } \text{Max}_{|z|=1}|P(z)| \text{ and } \text{Min}_{|z|=k}|P(z)|.$$

We first prove the following result which includes not only a refinement of inequality (10) analogous to (7) as a special case but also leads to some striking conclusions giving refinements and generalizations of other well known results.

Theorem 1. *If $P \in \Re_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| \leq k$, where $k \geq 1$, then for every $R \geq 1, 0 \leq t \leq 1$ and $|z| = 1$,*

$$|P(Rz) - P(z)| \leq \left(\frac{R^n - 1}{1 + k^\mu \psi(R, \mu, k)}\right)\left(\text{Max}_{|z|=1}|P(z)| - t\text{Min}_{|z|=k}|P(z)|\right), \tag{14}$$

where

$$\psi(R, \mu, k) := \frac{k + \lambda(R, \mu, k)}{1 + k\lambda(R, \mu, k)}, \tag{15}$$

and

$$\psi(R, \mu, k) := \left(\frac{R^n - 1}{R^n - 1}\right)\left(\frac{|a_\mu| k^n}{|a_0| - mt}\right) \leq 1 \quad \textit{with} \quad m = \text{Min}_{|z|=k}|P(z)|. \tag{16}$$

Instead of proving Theorem 1, we prove the following more general result which constitute a multifaced generalization of several well-known polynomial inequalities.

Theorem 2. *If $P \in \Re_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$, then for every $R > r \geq 1, 0 \leq t \leq 1$ and $|z| = 1$,*

$$|P(Rz) - P(rz)| \leq \left(\frac{R^n - r^n}{1 + k^\mu \phi_1(R, r, \mu, k)}\right)\left(\text{Max}_{|z|=1}|P(z)| - t\text{Min}_{|z|=k}|P(z)|\right), \tag{17}$$

where

$$\phi_1(R, r, \mu, k) := \frac{k + \lambda_1(R, r, \mu, k)}{1 + k\lambda_1(R, r, \mu, k)}, \tag{18}$$

and

$$\lambda_1(R, r, \mu, k) := \left(\frac{R^n - r^n}{R^n - r^n}\right)\left(\frac{|a_\mu| k^n}{|a_0| - tm}\right) \leq 1 \quad \textit{with} \quad m = \text{Min}_{|z|=k}|P(z)|. \tag{19}$$

Theorems 1 and 2, as stated above, have various interesting consequences. Here we mention a few of these.

Dividing the two sides of the inequality (14) by R-1 and making $R \to 1$, so that

$$\lambda(1,\mu,k) := \frac{\mu}{n}\frac{|a_\mu|k^\mu}{|a_0|-mt} \leq 1,$$

we immediately obtain the following interesting result which is a refinement as well as a generalization of inequality (10) and includes inequality (7) as a special case.

Corollary 1. *If $P \in \Re_{n,\mu}$ and P(z) does not vanish in the disk $|z| < k$, where $k \geq 1$, then for $0 \leq t \leq 1$ and $|z| = 1$,*

$$|P'(z)| \leq \left(\frac{n}{1+k^\mu \phi(1,\mu,k)}\right)\left(\mathrm{Max}_{|z|=1}|P(z)| - t\mathrm{Min}_{|z|=k}|P(z)|\right),$$

where

$$\phi(1,\mu,k) = \frac{k + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-mt}k^\mu}{1+\frac{\mu}{n}\frac{|a_\mu|}{|a_0|-mt}k^{\mu+1}}$$

and

$$m = \mathrm{Min}_{|z|=k}|P(z)|.$$

Remark 1. If we take $\mu = t = k = 1$ in Corollary 1 and use the fact that $\phi(1,1,1) = 1$, we get inequality (7). For $t = 0$, Corollary 1 reduces to (10). Inequality (14) also provides a refinement and a generalization of a result due to Aziz and Shah [6, Theorem 1]. Taking $t = 1$ and using the obvious inequality

$$|P(Rz)| \leq |P(Rz) - P(z)| + |P(z)|,$$

in Theorem 1, we get the following interesting result which is a generalization of inequality (8).

Corollary 2. *If $P \in \Re_{n,\mu}$ and P(z) does not vanish in the disk $|z| < k$, where $k \geq 1$, then for every $R \geq 1$,*

$$\mathrm{Max}_{|z|=R}|P(z)| \leq \frac{(R^n + k^\mu\phi(R,\mu,k))\left(\mathrm{Max}_{|z|=1}|P(z)| - (R^n-1)\mathrm{Min}_{|z|=k}|P(z)|\right)}{1+k^\mu\phi(R,\mu,k)}, \quad (20)$$

where $\phi(R,\mu,k)$ is defined by (15) and (16) with $t = 1$.

Remark 2. For $k = 1 = \mu$, inequality (20) reduces to (8).

Since $\lambda(R,\mu,k) \leq 1$, it can be easily seen that $\phi(R,\mu,k) \geq 1$ for $k \geq 1$.

Using these observations in (14), the following result is an immedaite consequence of Theorem 1 with $t = 1$ and Corollary 2.

Corollary 3. *If $P \in \Re_{n,\mu}$ and P(z) does not vanish in the disk $|z| < k$, where $k \geq 1$, then for every $R \geq 1$, and $|z| = 1$*

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{1 + k^n}\left(\text{Max}_{|z|=1}|P(z)| - \text{Min}_{|z|=k}|P(z)|\right) \tag{21}$$

and

$$\text{Max}_{|z|=R>1}|P(z)| \leq \frac{(R^n + k^\mu)\text{Max}_{|z|=1}|P(z)| - (R^n - 1)\text{Min}_{|z|=k}|P(z)|}{1 + k^\mu}. \tag{22}$$

Remark 3. If we divide the two sides of (21) by R and let $R \to 1$, it follows that if $P \in \Re_{\mu,n}$ and $P(z) \neq 0$ for $|z| < k$, where $k \geq 1$, then

$$\text{Max}_{|z|=1}|P^{'}(z)| \leq \frac{n}{1 + k^n}\left(\text{Max}_{|z|=1}|P(z)| - \text{Min}_{|z|=1}|P(z)|\right). \tag{23}$$

The result is sharp and equality in (20) holds for the polynomial $P(z) = \alpha\left(z^n + \beta k^n\right)^{\frac{n}{\mu}}$, where $\alpha \neq 0, |\beta| = 1$ and n is a multiple of μ. Inequality (22) is an interesting refinement of result due to Aziz and Shah [6] where as inequality (23) which was also proved by Dewan and Pulhta [17, Theorem 1. 4], Aziz and Rather [5, Inequality 14] earlier, provides an improvement of inequality (9).

The following corollary which is another interesting generalization of inequality (10) and a refinement of (9) can be deduced from Theorem 2 by dividing the both sides of (17) by R-r and making $R \to r$.

Corollary 4. *If $P \in \Re_{n,\mu}$ and P(z) does not vanish in the disk $|z| < k$, where $k \geq 1$, then for every $r \geq 1, 0 \leq t \leq 1$,*

$$\text{Max}_{|z|=r\geq 1}|P^{'}(z)| \leq \left(\frac{nr^{n-1}}{1 + k^\mu \phi_1(r,\mu,k)}\right)\left(\text{Max}_{|z|=1}|P(z)| - t\text{Min}_{|z|=k}|P(z)|\right),$$

where

$$\phi(r,\mu,k) = \frac{kr^{n-\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-mt}k^\mu}{r^{n-\mu} + \frac{\mu}{n}\frac{|a_\mu|}{|a_0|-mt}k^{\mu+1}}$$

and

$$m = \text{Min}_{|z|=k}|P(z)|.$$

If we take $\mu = 1$ and t=0, we get the following generalization of a result due to Govil, Rahman and Schmeisser [12].

Corollary 5. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n which does* *not vanish in* $|z| < k$ *where* $k \geq 1$*, then for every* $R \geq 1$*,*

$$\text{Max}_{|z|=R}|P^{'}(z) \leq nR^{n-1}\left(\frac{nR^{n-1}|a_0| + k^2|a_1|}{nR^{n-1}|a_0|(1+k^2) + 2k^2|a_1|}\right)\left(\text{Max}_{|z|=1}|P(z)|\right).$$

Next we prove the following theorem which provides an improvement as well as a generalization of the inequality (12).

Theorem 3. *If* $P \in \Re_{n,\mu}$ *and P(z) has no zeros in the disk* $|z| < k, k > 0$*, then for every fixed real or complex number* β *and* $0 \leq r \leq R \leq k$*,*

$$\text{Max}_{|z|=1}|P(Rz) - \beta P(rz)| \leq \left[(|\beta| + |1-\beta|)\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - |\beta|\right] \times \text{Max}_{|z|=r}|P(z)| - \left[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - 1\right]\text{Min}_{|z|=k}|P(z)|. \quad (24)$$

If we take $\beta = 1$, in Theorem 3, we immedaitely get the following improvement as well as a generalization of (12).

Corollary 6. *If* $P \in \Re_{n,\mu}$ *and P(z) has no zeros in the disk* $|z| < k, k > 0$*, then for* $0 \leq r \leq R \leq k$*,*

$$\text{Max}_{|z|=1}|P(Rz) - P(rz)| \leq \left[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - 1\right]\left[\text{Max}_{|z|=r}|P(z)| - \text{Min}_{|z|=k}|P(z)|\right] \quad (25)$$

and

$$\text{Max}_{|z|=1}|P(z)| \leq \left[\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right]^{\frac{n}{\mu}}\text{Max}_{|z|=r}|P(z)| - \left[\left(\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right)^{\frac{n}{\mu}} - 1\right]\text{Min}_{|z|=k}|P(z)|. \quad (26)$$

Both the estimates are sharp with in (25) and (26) for the polynomial $P(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

Finally as an application of Corollary 3 we prove the following theorem which not only extends and refines a result proved by Dewan and Bidkham [8] but in particular, also includes a result due to Aziz and Shah [6] as a special case.

Theorem 4. *If* $P \in \Re_{n,\mu}$ *and P(z) does not vanish in the disk* $|z| < k, k \geq 0$*, then for all* $\rho \geq 1$ *and* $0 \leq r \leq R \leq k$*,*

$$\text{Max}_{|z|=R}|P(\rho z) - P(z)|$$

$$\leq \frac{R^\mu(\rho^n - 1)}{r^\mu + k^\mu}\left[\frac{R^\mu + k^\mu}{r^\mu + k^\mu}\right]^{\frac{n}{\mu}-1}\left[\text{Max}_{|z|=r}|P(z)| - \text{Min}_{|z|=k}|P(z)|\right]. \quad (27)$$

Many interesting results can be deduced from Theorem 4 for different choices of the parameters. For example, dividing the both sides of (27) by $\rho - 1$ and making $\rho \to 1$, we immedaitely get the following result when $\mu = 1$ and $R = 1$.

Corollary 7. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n which does not vanish in the disk $|z| < k$, where $k \geq 1$, then for $0 \leq r \leq 1$,*

$$\text{Max}_{|z|=1}|P^{'}(z)| \leq \frac{n(1+k)^{n-1}}{(r+k)^n}\left[\text{Max}_{|z|=r}|P(z)| - \text{Min}_{|z|=k}|P(z)|\right]. \quad (28)$$

The result is the best possible and equality in (28) holds for the polynomial $P(z) = (z+k)^n$.

2. The Lemmas

For the proofs of these theorems we need the following lemmas.

Lemma 1. *If P(z) is a polynomial of degree n having all its zeros in $|z| \leq \rho$, where $\rho \geq 0$, then for every $R \geq r$, and $rR \geq \rho^2$,*

$$|P(Rz)| \geq \left(\frac{R+p}{r+p}\right)^n |P(rz)| \quad \textit{for} \quad |z| = 1. \quad (28)$$

Proof. Since all the zeros of P(z) lie in $|z| \leq \rho$, we write

$$P(z) = C\prod_{j=1}^{n}(z - k_j e^{i\theta_j}) \ \text{ where } \ k_j \leq \rho, \quad j = 1, 2, ..., n.$$

Now for $0 \leq \theta < 2\pi, R > r$ and $rR \geq k^2$, we clearly have

$$\left|\frac{Re^{i\theta} - k_j e^{i\theta_j}}{re^{i\theta} - k_j e^{i\theta_j}}\right| = \left[\frac{R^2 + k_j^2 - 2Rk_j \text{Cos}(\theta - \theta_j)}{r^2 + k_j^2 - 2rk_j \text{Cos}(\theta - \theta_j)}\right]^{\frac{1}{2}}$$

$$\geq \left[\frac{R + k_j}{r + k_j}\right] \geq \left[\frac{R+\rho}{r+\rho}\right], \quad j = 1, 2, ..., n.$$

Hence

$$\left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| = \prod_{j=1}^{n} \leq \frac{|Re^{i\theta} - k_j e^{i\theta_j}|}{|re^{i\theta} - k_j e^{i\theta_j}|} \geq \prod_{j=1}^{n}\left(\frac{R+\rho}{r+\rho}\right) = \left(\frac{R+\rho}{r+\rho}\right)^n,$$

for every point $e^{i\theta}$, $0 \leq \theta \leq 2\pi$. This implies

$$|P(Rz)| \geq \left(\frac{R+\rho}{r+\rho}\right)^n |P(rz)| \text{ for } |z|=1, \quad R>1 \text{ and } rR \geq \rho^2,$$

which completes the proof of Lemma 1. □

Lemma 2. *If $P \in \Re_{n,\mu}$ and P(z) does not vanish in the disk $|z| < k$, where $k \geq 1$, and $Q(z) = \overline{z^n P\left(\frac{k}{\bar{z}}\right)}$, then for $R \geq r \geq 1$ and $|z| = 1$,*

$$k^\mu \phi(R,r,\mu,k)|P(Rz) - P(rz)| \leq |Q(Rz) - Q(rz)|, \tag{29}$$

where

$$\phi(R,r,\mu,k) = \frac{k+\lambda(R,r,\mu,k)}{1+k\lambda(R,r,\mu,k)} \tag{30}$$

and

$$\phi(R,r,\mu,k) = \frac{R^n - r^n}{R^n - r^n}\left|\frac{a_n}{a_0}\right| k^\mu \leq 1. \tag{31}$$

Proof. For $R = r$, there is nothing to prove. Henceforth we assume that $R > r$. Since the polynomial P(z) has all its zeros in $|z| \geq k$, the polynomial $F(z) = P(kz)$ does not vanish i $|z| < 1$.

This implies that the poynomial $G(r) = z^n F\left(\frac{1}{z^-}\right)^-$, has all its zeros in $|z| \leq 1$ and $|G(z)| = |F(z)|$ for $|z| = 1$. Therefore, the function $\frac{G(z)}{F(z)}$ is analytic in $|z| \leq 1$ and

$$\left|\frac{G(z)}{F(z)}\right| = 1 \quad \text{for} \quad |z| = 1.$$

By maximum modulus principle, it follows that $|G(z)| \leq |F(z)|$ for $|z| \leq 1$. Replacing z by $\frac{1}{z}$, we obtain

$$|F(z)| \leq |G(z)| \text{ for } |z| \geq 1\,. \tag{32}$$

Hence for every real or a complex number β, with $|\beta| > 1$, the polynomial $H(z) = F(z) - \beta G(z)$ has all its zeros in $|z| \leq 1$. Because if this is not the case, then there is a point $z = z_0$ with $|z_0| > 1$ such that $H(z_0) = 0$. This gives $F(z_0) = \beta G(z_0)$. Since $G(z_0) \neq 0$ as $|z_0| > 1$ and $|\beta| > 1$, we get

$$|F(z_0)| = |\beta||G(z_0)| > |G(z_0)| \quad |z_0| > 1,$$

which is clearly a contardiction to (32). Now applying Lemma 1 with $\rho = 1$ to the polynomial $H(z)$ having all its zeros in $|z| \leq 1$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$.

$$\left|H(Re^{i\theta})\right| \geq \left(\frac{R+1}{r+1}\right)^n |H(re^{i\theta})|. \tag{33}$$

Since $H(re^{i\theta}) \neq 0$ for every $R > r \geq 1$, $0 \leq \theta < 2\pi$ and $R+1 > r+1$, it follows from (33) that

$$\left|H(Re^{i\theta})\right| \geq \left(\frac{r+1}{R+1}\right)^n |H(re^{i\theta})| \geq |H(re^{i\theta})|,$$

for every $R > r > 1$ and $o \leq \theta < 2\pi$. This gives

$$|H(rz)| < H(Rz) \quad \text{for} \quad |z| = 1 \quad \text{and} \quad R > rge1.$$

Using Rouche's Theorem and noting that all the zeros of $H(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, we conclude that the polynomial

$$\psi(z) = H(Rz) - H(rz) = [F(Rz) - F(rz)] - \beta\,[G(Rz) - G(rz)], \tag{34}$$

has all its zeros in $|z| < 1$ for every real or complex number β with $|\beta| > 1$ ang $R > r \geq 1$. This implies

$$|F(Rz) - F(rz)| \leq |G(Rz) - G(rz)|, \tag{35}$$

for $|z| \geq 1$ and $R \geq r \geq 1$, if inequality (35) is not true, then there exists a point $z = w$ with $|w| \geq 1$, such that

$$|F(Rw) - F(rw)| > |G(Rw) - G(rw)|.$$

But all the zeros $G(z)$ lie in $|z| \leq 1$, therefore, it follows (as in the case of $H(z)$) that all the zeros of the polynomial $G(Rz) - G(rz)$ lie in $|z| < 1$ for every $R > r \geq 1$. Hence $G(Rw) - G(rw) \neq 0$ with $|w| \geq 1$. We take

$$\beta = \frac{F(Rw) - F(rw)}{G(Rw) - G(rw)}.$$

Then β is a well defined real or a complex number with $|\beta| > 1$ and with this choice of β, from (34) we obtain $\psi(w) = 0$, where $|w| \geq 1$. This contradicts the fact that all the zeros of $\psi(z)$ lie in $|z| < 1$. Thus

$$|F(Rz) - F(rz)| \leq |G(Rz) - G(rz)| \quad \text{for} \quad |z| \geq 1 \quad \text{and} \quad R \geq r \geq 1.$$

Replacing $F(z)$ by $P(kz)$ and $G(z)$ by $z^n \overline{P\left(\frac{k}{\bar{z}}\right)}$,

$$\begin{aligned} |P(Rkz) - P(rkz)| &\leq \left| R^n z^n \overline{P\left(\frac{k}{R\bar{z}}\right)} - r^n z^n \overline{P\left(\frac{k}{r\bar{z}}\right)} \right| \\ &= \left| R^n \overline{P\left(\frac{kz}{R}\right)} - r^n \overline{P\left(\frac{kz}{r}\right)} \right|. \end{aligned} \tag{36}$$

for $|z| = 1$ and $R > r \geq 1$.

Now $P \in \Re_{n,\mu}$ implies that

$$P(Rkz) - P(rkz) = a_n(R^n - r^n)k^n z^n + ... + a_\mu(R^\mu - r\mu)k^\mu z^\mu$$

$$= k^{\mu}z^{\mu}\left\{a_n(R^n-r^n)(kz)^{n-\mu}+...+a_{\mu}(R^n-r^{\mu})\right\}.$$

Since all the zeros of

$$G(Rz)-G(rz)=R^nz^n\overline{P\left(\frac{k}{R\bar{z}}\right)}-r^nz^n\overline{P\left(\frac{k}{r\bar{z}}\right)},$$

lie in $|z|<1$ for every $R>r\geq 1$, it follows that the polynomial

$$R^nP\left(\frac{kz}{R}\right)-r^nP\left(\frac{kz}{r}\right)=z^n\overline{G\left(\frac{R}{\bar{z}}\right)}-z^n\overline{G\left(\frac{r}{\bar{z}}\right)},$$

does not vanish in $|z|\leq 1$. Hence the function

$$S(z)=\frac{P(Rkz)-P(rkz)}{z^{\mu}\left[R^nP\left(\frac{kz}{R}\right)-r^nP\left(\frac{kz}{r}\right)\right]},$$

$$S=\frac{k^{\mu}\left\{a_n(R^n-r^n)(kz)^{n-\mu}+...+a_{\mu}(R^n-r^{\mu})\right\}}{\left\{R^nP\left(\frac{kz}{R}\right)-r^nP\left(\frac{kz}{r}\right)\right\}}, \qquad (37)$$

is analytic in $|z|\leq 1$, and by (36),

$$S(z)=\left|\frac{P(Rkz)-P(rkz)}{z^{\mu}\left[R^nP\left(\frac{kz}{R}\right)-r^nP\left(\frac{kz}{r}\right)\right]}\right|=\left|\frac{P(Rkz)-P(rkz)}{\left[R^nP\left(\frac{kz}{R}\right)-r^nP\left(\frac{kz}{r}\right)\right]}\right|\leq 1, \qquad (38)$$

for $|z|=1, R>r\geq 1$. A direct application of the maximum modulus principle shows that

$$S(z)\leq 1 \quad \text{for} \quad |z|\leq 1.$$

In particular $|S(0)|\leq 1$,which implies by (37) that

$$\frac{R^{\mu}-r^{\mu}}{R^n-r^n}\left|\frac{a_{\mu}}{a_0}\right|k^{\mu}\leq 1 \quad \text{for} \quad R>r\geq 1, \qquad (39)$$

This proves (31). To establish (29), we apply Schwartz's Lemma [23, p. 212] to the function $S(z)$ to conclude that

$$|S(z)|=\frac{|z|+\frac{R^{\mu}-r^{\mu}}{R^n-r^n}\left|\frac{a_{\mu}}{a_0}\right|k^{\mu}}{1+\frac{R^{\mu}-r^{\mu}}{R^n-r^n}\left|\frac{a_{\mu}}{a_0}\right|k^{\mu}|z|} \quad \text{for} \quad |z|\leq 1.$$

With the help of (31) and (37), this implies that

$$\left|\frac{P(Rkz)-P(rkz)}{z^{\mu}\left[R^nP\left(\frac{kz}{R}\right)-r^nP\left(\frac{kz}{r}\right)\right]}\right|\leq\frac{|z|+\lambda(R,r,\mu,k)}{1+|z|\lambda(R,r,\mu,k)},$$

for $|z| \leq 1$. We take $z = e^{\frac{i\theta}{k}}$, $0 \leq \theta < 2\pi$, so that $|z| = \frac{1}{k} \leq 1$ and we get

$$k^{\mu} \left| \frac{P(Re^{i\theta}) - P(re^{i\theta})}{R^n \left[R^n P\left(\frac{e^{i\theta}}{R}\right) - r^n P\left(\frac{e^{i\theta}}{r}\right)\right]} \right| \leq \left\{ \frac{1 + k\lambda(R,r,\mu,k)}{k + \lambda(R,r,\mu,k)} \right\}. \tag{40}$$

By hypothesis, $Q(z) = z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$, therefore, for every $R > r$ and $O \leq \theta < 2\pi$, we have

$$\left|Q(Re^{i\theta}) - Q(re^{i\theta})\right| = \left| R^n e^{in\theta} \overline{P\left(\frac{e^{i\theta}}{R}\right)} - r^n e^{in\theta} \overline{P\left(\frac{e^{i\theta}}{r}\right)} \right|$$
$$= \left| R^n P\left(\frac{e^{i\theta}}{R}\right) - r^n P\left(\frac{e^{i\theta}}{r}\right) \right|.$$

Using this in (40), it follows that for every $R > r \geq 1$ and $|z| = 1$,

$$k^{\mu} \left| \frac{P(Rz) - P(rz)}{Q(Rz) - Q(rz)} \right| \leq \frac{1}{\phi(R,r,\mu,k)}, \tag{41}$$

where $\phi(R,r,\mu,k)$ is defined by (30). The inequality (41) is equivalent to (29) and this completets the proof of Lemma 2. □

Lemma 3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq \rho$, where $\rho \leq 1$, then for every $R \geq r \geq 1$ and $|z| = 1$,*

$$|P(Rz) - P(rz)| \geq \frac{R^n - r^n}{\rho^n} \mathrm{Min}_{|z|=\rho} |P(z)|. \tag{42}$$

Proof. The result is obvious for $R = r$. So we assume $R > r \geq 1$. Let $m = \mathrm{Min}_{|z|=\rho}|P(z)|$. Since all the zeros of $P(z)$ lie in $|z| \leq \rho$, where $\rho \leq 1$, therefore, all the zeros of the polynomial $F(z) = P(\rho z)$ lie in $|z| \leq 1$, and

$$\mathrm{Min}_{|z|=1}|F(z)| = \mathrm{Min}_{|z|=1}|P(\rho z)| = \mathrm{Min}_{|z|=\rho}|P(z)| = m.$$

This implies $m|z|^n = m \leq |F(z)|$ for $|z| = 1$. We first show that the polynomial $H(z) = F(z) + \alpha m z^n$ has all its zeros in $|z| \leq 1$ for every real or a complex number α with $|\alpha| < 1$. This is obvious if $m = 0$, that is if $F(z)$ has a zero on $|z| = 1$. We now assume that all the zeros of $F(z)$ lie in $|z| < 1$, so that $m > 0$. This implies for $|z| = 1$ and $|\alpha| < 1$,

$$|m\alpha z^n| = m|\alpha||z|^n = m|\alpha| < m \leq |F(z)|.$$

A direct application of Rouche's Theorem shows that the polynomial $H(z) = F(z) + \alpha m z^n$ of degree n has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| < 1$. Applying Lemma 1 to the polynomial $H(z)$ with $\rho = 1$

and noting that $H(Rz) \neq 0$ for $|z| = 1$ and $R > r \geq 1$, we get

$$|H(Rz)| > \left(\frac{r+1}{R+1}\right)^n |H(Rz)| \geq H(rz) \quad \text{for} \quad |z| = 1. \tag{43}$$

Since all the zeros of $H(Rz)$ lie in $|z| < \frac{1}{R} < 1$, by Rouche's Theorem again it follows from (43) that all the zeros of the polynomial

$$T(z) = H(Rz) - H(rz) = F(Rz) - F(rz) + \alpha\ m\Big(R^n - r^n\Big)z^n \tag{44}$$

lie in $|z| < 1$ for every real or a complex number α with $|\alpha| < 1$ and $R > r \geq 1$. This gives

$$\Big|F(Rz) - F(rz)\Big| \geq m\Big(R^n - r^n\Big)|z^n|, \tag{45}$$

for $|z| \geq 1$ and $R > r \geq 1$. If the inequality (45) is not true, then thre is a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\Big|F(Rz_0) - F(rz_0)\Big| < m\Big(R^n - r^n\Big)|z_0^n|, \quad R > r \geq 1.$$

We choose

$$\alpha = \frac{F(Rz_0) - F(rz_0)}{m(R^n - r^n)z_0^n},$$

then clearly $|\alpha| < 1$ and with this choice of α, we get

$$T(z_0) = F(Rz_0) - F(rz_0) - \alpha\ m(R^n - r^n)z_0^n = 0,$$

where $|z_0| \geq 1$. This is a contradiction to (44). Hence the inequality (45) is established. Replacing $F(z)$ by $P(\rho z)$ in (45), we obtain

$$|P(R\rho z) - P(r\rho z)| \geq m(R^n - r^n)|z^n|,$$

for every $R > r \geq 1$ and $|z| \geq 1$. Taking in particular, $z = \frac{e^{i\theta}}{\rho}$, where $\rho \leq 1$, then $|z| = \frac{1}{\rho} \geq 1$ and we get

$$\left|P\Big(Re^{i\theta}\Big) - P\Big(re^{i\theta}\Big)\right| \geq \frac{m(R^n - r^n)}{\rho^n}, \tag{46}$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. The inequality (46) is equivalent to the desired result and the proof of Lemma 3 is complete. □

Lemma 4. *If $P \in \Re_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$, and $Q(z) = z^n \overline{P\Big(\frac{k}{\bar{z}}\Big)}$, then for $R \geq r \geq 1$ and $|z| = 1$,*

$$k^\mu \phi_1(R, r, \mu, k)|P(Rz) - P(rz)| \leq |Q(Rz) - Q(rz)| - (R^n - r^n)t\ m, \tag{47}$$

where

$$\phi_1(R,r,\mu,k) = \frac{k + \lambda_1(R,r,\mu,k)}{1 + k\lambda_1(R,r,\mu,k)}$$

and

$$\lambda_1(R,r,\mu,k) = \left(\frac{R^n - r^n}{R^n - r^n}\right)\left(\frac{|a_\mu|k^\mu}{|a_0| - t\ m}\right) \leq 1 \quad \textit{with} \ \ m = \text{Min}_{|z|=k}|P(z)|. \tag{48}$$

Proof. For $R = r$, the result is obvious. So we assume $R > r$. By hypothesis, the polynomial $P(z)$ has all its zeros in $|z| \geq k \geq 1$ and $m = \text{Min}_{|z|=k}|P(z)|$, therefore, $m \leq |P(z)|$ for $|z| = k$. We first show that for any given real or a complex number α with $|\alpha| \leq 1$, the polynomial $F(z) = P(z) - \alpha\ m$ does not vanish in $|z| < k$. This is clear if $m = 0$, that is if $P(z)$ has a zero on $|z| = k$. We now suppose that all the zeros of $P(z)$ lie in $|z| > k$, then clearly $m > 0$ so that $\frac{m}{P(z)}$ is analytic in $|z| \leq k$ and

$$\left|\frac{m}{P(z)}\right| \leq 1 \quad \text{for} \quad |z| = k.$$

Since $\frac{m}{P(z)}$ is not a constant, by the maximum modulus principle, it follows that

$$m < |P(z)| \quad \text{for} \quad |z| < k. \tag{49}$$

Now if $F(z) = P(z) - \alpha\ m$ has a zero in $|z| < k$, say at $z = z_0$ with $|z_0| < k$, then $P(z_0) - \alpha\ m = F(z_0) = 0$. This gives

$$|P(z_0)| = |\alpha\ m| = |\alpha| m \leq m\,, \quad \text{where} \ \ |z_0| < k,$$

which contradicts (49). Hence we conclude that in any case, the polynomial $F(z) = P(z) - \alpha\ m$ does not vanish in $|z| < k$, where $k \geq 1$, for every real or a complex number α with $|\alpha| \leq 1$. Applying Lemma 2 to the polynomial

$$F(z) = P(z) - \alpha\ m = (a_0 - \alpha\ m) + \sum_{j=\mu}^{n} a_j z^j,$$

we get for every real or a complex number α with $|\alpha| \leq 1$, $R \geq r \geq 1$ and $|z| = 1$,

$$k^\mu \phi(R,r,\mu,k)\Big|P(Rz) - P(rz)\Big| \leq \Big|Q(Rz) - Q(rz) - \bar{\alpha}(R^n - r^n)mz^n\Big|, \tag{50}$$

where

$$\phi(R,r,\mu,k) = \frac{k + \lambda}{1 + k\lambda}$$

and

$$\lambda = \lambda(R, r, \mu, k) = \frac{R^\mu - r^\mu}{R^n - r^n}\frac{|a_\mu|k^\mu}{|a_0 - \alpha\ m|} \leq 1. \tag{51}$$

Using $|a_0| - |\alpha|m \leq |a_0 - \alpha\ m|$. and $k \geq 1$, it can be easily verified after a short calculation that

$$\phi_1(R, r, \mu, k) = \frac{k + \left[\frac{R^\mu - r^\mu}{R^n - r^n}\right]\frac{|a_\mu|k^\mu}{|a_0| - |\alpha|m}}{1 + k\left[\frac{R^\mu - r^\mu}{R^n - r^n}\right]\frac{|a_\mu|k^\mu}{|a_0| - |\alpha|m}} \leq \frac{k + \left[\frac{R^\mu - r^\mu}{R^n - r^n}\right]\frac{|a_\mu|k^\mu}{|a_0 - \alpha m|}}{1 + k\left[\frac{R^\mu - r^\mu}{R^n - r^n}\right]\frac{|a_\mu|k^\mu}{|a_0 - \alpha m|}} = \phi(R, r, \mu, k). \tag{52}$$

Since all the zeros of $Q(z) = z^n\overline{P\left(\frac{1}{\bar z}\right)}$ lie in $|z| \leq \left(\frac{1}{k}\right) \leq 1$ and

$$\begin{aligned}\text{Min}_{|z|=\frac{1}{k}}|Q(z)| &= \text{Min}_{|z|=\frac{1}{k}}\left|z^n\overline{P\left(\frac{1}{\bar z}\right)}\right| = \text{Min}_{|z|=\frac{1}{k}}\left|\frac{z^n}{k^n}P\left(\frac{k}{\bar z}\right)\right| \\ &= \frac{1}{k^n}\text{Min}_{|z|=1}|P(kz)| = \frac{1}{k^n}\text{Min}_{|z|=k}|P(z)| = \frac{m}{k^n}.\end{aligned}$$

We conclude that by Lemma 3 (with $P(z)$ replaced by $Q(z)$ and ρ by $\frac{1}{k}$) that

$$|Q(Rz) - Q(rz)| \geq \frac{(R^n - r^n)}{\frac{1}{k^n}}\text{Min}_{|z|=\frac{1}{k}}|Q(z)| = m(R^n - r^n), \tag{53}$$

for $|z| = 1$ and $R \geq r \geq 1$. Choosing argument of $\alpha, |\alpha| \leq 1$, on teh R.H.S. of (50) such that for $|z| = 1$,

$$\left|Q(Rz) - Q(rz) - \bar\alpha(R^n - r^n)mz^n\right| = \left|Q(Rz) - Q(rz) - |\alpha|m(R^n - r^n)\right|$$

(which is possible by (53)), it follows from (50) by using (52) that for every $R \geq r \geq 1, |\alpha| \leq 1$ and $|z| = 1$,

$$k^\mu\phi_1(R, r, \mu, k)\left|P(Rz) - P(rz)\right| \leq \left|Q(Rz) - Q(rz) - |\alpha|m(R^n - r^n)\right|,$$

where

$$\phi_1(R, r, \mu, k) = \frac{k + \lambda_1}{1 + k\lambda_1},$$

and

$$\lambda_1 = \lambda_1(R, r, \mu, k) = \frac{R^\mu - r^\mu}{R^n - r^n}\frac{|a_\mu|k^\mu}{|a_0| - |\alpha|m}.$$

Since (51) is true for all α with $|\alpha| \leq 1$ and by (49), $m < |P(0)| = |\alpha_0|$, we can choose argument of α in (51) such that $|a_0 - \alpha m| = |\alpha_0| - |\alpha|m$. For this choice of the argument of α, we get from (51) that

$$\lambda_1(R,r,\mu,k) = \lambda(R,r,\mu,k) \leq 1. \tag{55}$$

The inequalities (54) and (55) are equivalent to (47) and (48) respectively with $t = |\alpha|$, where $0 \leq t \leq 1$. This completes the proof of Lemma 4. □

We also need the following lemma which is a special case of a result due to Govil and Rahman [11, Lemma 10] (see also [4]).

Lemma 5. *If $P(z)$ is a polynomial of degree n, then for $|z| = 1$,*

$$|P^{'}(z)| + |Q^{'}(z)| \leq n\text{Max}_{|z|=1}|P(z)|,$$

where $Q(z) = z^n \overline{P\left(\frac{k}{\bar{z}}\right)}$.

We use Lemma 5 to prove the following result which we need for the proof of Theorem 2 and is also of independent interest.

Lemma 6. *If $P(z)$ is a polynomial of degree n, then for every $R \geq r \geq 1$ and $|z| = 1$,*

$$|P(Rz) - P(rz)| + |Q(Rz) - Q(rz)| \leq (R^n - r^n)\text{Max}_{|z|=1}|P(z)|\,,$$

where $Q(z) = z^n \overline{P\left(\frac{k}{\bar{z}}\right)}$.

Proof. For every real or complex number α, with$|\alpha| = 1$, by Lemma (5), we have,

$$|P^{'}(z) + \alpha Q^{'}(z)| \leq n\text{Max}_{|z|=1}|P(z)|. \tag{56}$$

Applying inequality (2) to the polynomial $P^{'}(z) + \alpha Q^{'}(z)$ which is of degree $n-1$ and using (56), we obtain for all $t \geq 1, 0 \leq \theta \leq 2\pi$ and $|\alpha| = 1$,

$$\left|P^{'}\left(te^{i\theta}\right) + \alpha Q^{'}\left(te^{i\theta}\right)\right| \leq nt^{n-1}\text{Max}_{|z|=1}|P(z)|. \tag{57}$$

Choosing argument of α on the L. H. S. of (57) suitably, we get

$$\left|P^{'}\left(te^{i\theta}\right)\right| + \left|Q^{'}\left(te^{i\theta}\right)\right| \leq nt^{n-1}\text{Max}_{|z|=1}\left|P(z)\right|. \tag{58}$$

for all $t \geq 1$ and $0 \leq \theta < 2\pi$. Hence for every $R \geq r \geq 1$ and $0 \leq \theta < 2\pi$, we have with the help of (58),

$$\left|P\left(Re^{i\theta}\right) - P\left(re^{i\theta}\right)\right| + \left|Q\left(Re^{i\theta}\right) - Q\left(re^{i\theta}\right)\right|$$
$$= \left|\int_r^R e^{i\theta}P^{'}\left(te^{i\theta}\right)dt\right| + \left|\int_r^R e^{i\theta}Q^{'}\left(te^{i\theta}\right)dt\right|$$

$$\leq \int_r^R \left|P^{'}\left(te^{i\theta}\right)\right|dt + \int_r^R \left|Q^{'}\left(te^{i\theta}\right)\right|dt = \left\{\int_r^R \left|P^{'}\left(te^{i\theta}\right)\right| + \left|Q^{'}\left(te^{i\theta}\right)\right|\right\} dt$$

$$\leq \left\{\int_r^R nt^{n-1}dt\right\} \text{Max}_{|z|=1}|P(z)| = (R^n - r^n)\text{Max}_{|z|=1}|P(z)|,$$

which is equivalent to the desired result. □

3. Proofs of Theorems

Proof of Theorem 1. This follows by takin $r = 1$ in Theorem 2. □

Proof of Theorem 2. By hypothesis $P \in \Re_{n,\mu}$ and $P(z) \neq 0$ for $|z| < k$, where $k \geq 1$, therefore, by Lemma 4, for every $R \geq r \geq 1, 0 \leq t \leq 1$, and $|z| = 1$, we have

$$k^{\mu}\phi_1(R, r, \mu, k)|P(Rz) - P(rz)| \leq |Q(Rz) - Q(rz)| - (R^n - r^n)\, t\text{Min}_{|z|=k}|P(z)|, \quad (59)$$

where $k^{\mu}\phi_1(R, r, \mu, k)$ is defined by (18) and (48). Also be Lemma 6, we get

$$|P(Rz) - P(rz)| + |Q(Rz) - Q(rz)| \leq (R^n - r^n)\, t\text{Min}_{|z|=k}|P(z)|, \quad (60)$$

for $|z| = 1$ and for every $R \geq r \geq 1$. Inequality (59) with the help of (60) yeilds

$$\{1 + k^{\mu}\phi_1(R, r, \mu, k)\}\, |P(Rz) - P(rz)| \leq (R^n - r^n)\left\{\text{Max}_{|z|=1}|P(z)| - t\text{Min}_{|z|=k}|P(z)|\right\},$$

for every $R \geq r \geq 1, 0 \leq t \leq 1$, and $|z| = 1$, which is equivalent to the inequality (17). Since the inequality (19) is same as inequality (48) of Lemma 4, the proof of Theorem 2 is complete. □

Proof of Theorem 3. Since $P \in \Re_{n,\mu}$ and $P(z) \neq 0$ for $|z| < k$, where $k > 0$, the polynomial $F(z) = P(sz)$ has no zero in $|z| < \frac{k}{s}, s > 0$ and $F \epsilon \Re_{n,\mu}$. Hence for $0 \leq s \leq k$, it follows from inequality (23) of Remark 3 (with k replaced by $\frac{k}{s} \geq 1$) that

$$\text{Max}_{|z|=1}|F(z)| = \text{Max}_{|z|=1}|P^{'}(sz)| \leq \frac{ns^{\mu}}{s^{\mu} + k^{\mu}}\left\{\text{Max}_{|z|=1}|P(sz)| - \text{Min}_{|z|=\frac{k}{s}}|P(sz)|\right\}.$$

This implies

$$\text{Max}_{|z|=1}|P^{'}(sz)| \leq \frac{ns^{\mu-1}}{s^{\mu} + k^{\mu}}\left\{\text{Max}_{|z|=1}|P(z)| - \text{Min}_{|z|=k}|P(z)|\right\}. \quad (61)$$

For $0 \le r \le R \le k$ and $0 \le \theta < 2\pi$, we have

$$P\left(Re^{i\theta}\right) - P\left(re^{i\theta}\right) = \int_r^R e^{i\theta} P'\left(se^{i\theta}\right) ds,$$

which gives

$$P\left(Re^{i\theta}\right) - \beta P\left(re^{i\theta}\right) = (1-\beta)\, P\left(re^{i\theta}\right) \int_r^R e^{i\theta} P'\left(se^{i\theta}\right) ds,$$

where β is any fixed real or a complex number. Hence for every $\theta, 0 \le \theta < 2\pi$ and $0 \le r \le R \le k$,

$$\left|P\left(Re^{i\theta}\right) - \beta P\left(re^{i\theta}\right)\right| \le |(1-\beta)| \left|P\left(re^{i\theta}\right)\right| + \int_r^R \left|P'\left(se^{i\theta}\right)\right| ds,$$

from which it follows that

$$\begin{aligned} &\text{Max}_{|z|=1} \left|P(Rz) - \beta P(rz)\right| \\ &\qquad \le |(1-\beta)|\, \text{Max}_{|z|=1} \left|P(rz)\right| + \int_r^R \text{Max}_{|z|=1} \left|P'(sz)\right| ds, \quad (62)\end{aligned}$$

Using (61) in (62), we get

$$\begin{aligned} \text{Max}_{|z|=1} \left|P(Rz) - \beta P(rz)\right| &\le |(1-\beta)|\, \text{Max}_{|z|=1} \left|P(rz)\right| \\ &+ \int_r^R \frac{ns^{\mu-1}}{s^\mu + k^\mu} \left\{\text{Max}_{|z|=1}|P(z)| - \text{Min}_{|z|=k}|P(z)|\right\} ds. \quad (63)\end{aligned}$$

Now clearly

$$\begin{aligned} \text{Max}_{|z|=s}|P(z)| &= \text{Max}_{|z|=1|}P(sz)| \\ &\le \text{Max}_{|z|=1}\left\{|P(sz)| - \beta P(rz)| + |\beta||P(rz)|\right\} \\ &\le \text{Max}_{|z|=1}|P(sz)| - \beta P(rz)| + |\beta|\text{Max}_{|z|=1}|P(rz)|. \quad (64)\end{aligned}$$

The inequality (63) gives with the help of (64) that

$$\text{Max}_{|z|=1}|P(Rz)| - \beta P(rz)| \le |(1-\beta)|\text{Max}_{|z|=1}|P(rz)| + n\int_r^R \frac{s^{\mu-1}}{s^\mu + k^\mu} \times$$

$$\left\{\text{Max}_{|z|=1}|P(sz) - \beta P(rz)| + |\beta|\text{Max}_{|z|=1}|P(rz)| - \text{Min}_{|z|=k}|P(z)|\right\} ds, \quad (65)$$

If we denote the right hand side of the inequality (65) by $\phi(R)$, then we have

$$\phi^{'}(R) = \frac{nR^{\mu-1}}{R^{\mu}+k^{\mu}}\times \left\{\text{Max}_{|z|=1}|P(Rz)-\beta P(rz)| + |\beta|\text{Max}_{|z|=1}|P(rz)| - \text{Min}_{|z|=k}|P(z)|\right\}, \quad (66)$$

(65) can be written as

$$\text{Max}_{|z|=1}|P(Rz)-\beta P(rz)| \leq \phi(R). \quad (67)$$

With the help of (67), the inequality (66) implies for $0 \leq r \leq R \leq k$ that

$$\phi^{'}(R) = \frac{nR^{\mu-1}}{R^{\mu}+k^{\mu}}\left\{\phi(R) + |\beta|\text{Max}_{|z|=1}|P(rz)| - \text{Min}_{|z|=k}|P(z)|\right\} \leq 0. \quad (68)$$

Multiplying the two sides of (68) by $\left(R^{\mu}+k^{\mu}\right)^{\frac{-n}{\mu}}$, we obtain

$$\frac{d}{dR}\left[\left(R^{\mu}+k^{\mu}\right)^{\frac{-n}{\mu}}\left\{\phi(R) + |\beta|\text{Max}_{|z|=1}|P(rz)| - \text{Min}_{|z|=k}|P(z)|\right\}\right] \leq 0, \quad (69)$$

for $0 \leq r \leq R \leq k$. Inequality (69) implies that

$$\left(R^{\mu}+k^{\mu}\right)^{\frac{-n}{\mu}}\left\{\phi(R) + |\beta|\text{Max}_{|z|=1}|P(rz)| - \text{Min}_{|z|=k}|P(z)|\right\}$$

is a decreasing function of R in $[0,k]$. Hence for $0 \leq r \leq R \leq k$, we have

$$\left\{\phi(R) + |\beta|\text{Max}_{|z|=1}|P(rz)| - \text{Min}_{|z|=k}|P(z)|\right\}/\left(R^{\mu}+k^{\mu}\right)^{\frac{n}{\mu}} \leq \left\{\phi(r) + |\beta|\text{Max}_{|z|=1}|P(rz)| - \text{Min}_{|z|=k}|P(z)|\right\}/\left(r^{\mu}+k^{\mu}\right)^{\frac{n}{\mu}},$$

which gives with the help of (67) and the fact that

$$\phi(r) = |1-\beta|\text{Max}_{|z|=1}|P(rz)|, \text{Max}_{|z|=1}|P(Rz)-\beta P(rz)|$$

$$\leq \left\{\frac{R^{\mu}+k^{\mu}}{r^{\mu}+k^{\mu}}\right\}^{\frac{n}{\mu}}\left\{(|1-\beta|+|\beta|)\,\text{Max}_{|z|=1}|P(rz) - \text{Min}_{|z|=k}|P(z)|\right\} -|\beta|\text{Max}_{|z|=1}|P(rz)| + \text{Min}_{|z|=k}|P(z)|.$$

Equivalently

$\text{Max}_{|z|=1}|P(Rz)-\beta P(rz)|$

$$\leq \left[(|\beta| + |1-\beta|) \left\{ \frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right\}^{\frac{n}{\mu}} - |\beta| \right] \mathrm{Max}_{|z|=r} |P(z)| \\ - \left[\left\{ \frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right\}^{\frac{n}{\mu}} - 1 \right] \mathrm{Min}_{|z|=k} |P(z)|,$$

for $0 \leq r \leq R \leq k$ which is (24) and this completes the proof of Theorem 3. □

Proof of Theorem 4. By hypothesis $P \in \Re_{n,\mu}$ and $P(z)$ does not vanish in $|z| < k$, where $k \geq 0$, therefore, the polynomial $F(z) = P(Rz)$ does not vanish in $|z| < \frac{k}{R}, R > 0$ and $F \epsilon \Re_{n,\mu}$. Hence for $0 < R \leq k$, it follows from inequality (21) of Corollary 3 (with k replaced by $\frac{k}{R} \geq 1$) that for every $\rho \geq 1$,

$$max_{|z|=1} |F(\rho z) - F(z)| \\ \leq \frac{(\rho^n - 1)}{1 + \left(\frac{k}{R}\right)^\mu} \left\{ \mathrm{Max}_{|z|=1} |F(z)| - \mathrm{Min}_{|z|=\frac{k}{R}} |F(z)| \right\}. \quad (70)$$

Replacing $F(z)$ by $P(Rz)$ and noting that

$$\mathrm{Max}_{|z|=1} |F(z)| = \mathrm{Max}_{|z|=1} |P(Rz)| = \mathrm{Max}_{|z|=R} |P(z)|$$

and

$$\mathrm{Min}_{|z|=\frac{k}{R}} |F(z)| = \mathrm{Min}_{|z|=\frac{k}{R}} |P(Rz)| = \mathrm{Min}_{|z|=k} |P(z)|,$$

from (70) it follows that

$$\mathrm{Max}_{|z|=1} |P(R\rho z) - P(Rz)| \\ \leq \frac{R^\mu(\rho^n - 1)}{R^\mu + k^\mu} \left\{ \mathrm{Max}_{|z|=R} |P(z)| - \mathrm{Min}_{|z|=k} |P(z)| \right\}, \quad (71)$$

for $\rho \geq 1$ and $0 \leq R \leq k$. Now if $0 \leq r \leq R \leq k$, then by inequality (26) of Corollary 6, we have

$$\mathrm{Max}_{|z|=R} |P(z)| \leq \left\{ \frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right\}^{\frac{n}{\mu}} \\ \times \left\{ \mathrm{Max}_{|z|=r} |P(z)| - \mathrm{Min}_{|z|=k} |P(z)| \right\} + \mathrm{Min}_{|z|=k} |P(z)|. \quad (72)$$

Using (72) in (71), we obtain

$$\mathrm{Max}_{|z|=R} |P(\rho z) - P(z)| = \mathrm{Max}_{|z|=1} |P(R\rho z) - P(Rz)| \\ \leq \frac{R^\mu(\rho^n - 1)}{R^\mu + k^\mu} \left\{ \frac{R^\mu + k^\mu}{r^\mu + k^\mu} \right\}^{\frac{n}{\mu}-1} \left\{ \mathrm{Max}_{|z|=r} |P(z)| - \mathrm{Min}_{|z|=k} |P(z)| \right\},$$

for $0 \leq r \leq R \leq k$ and $\rho \geq 1$, which is (27) and this completes the proof of Theorem 4. □

Acknowledgments

The author is extremely thankful to the University of Kashmir, Srinagar, for providing financial assistance concerning the page charges of this paper.

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