International Journal of Pure and Applied Mathematics
Volume 41 No. 5 2007, 747-752

# A NOTE ON MONOTONE ITERATIVE TECHNIQUE FOR FUNCTIONAL DIFFERENTIAL SYSTEMS WITH RETARDATION AND ANTICIPATION

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**Abstract:** Another method of monotone iterative technique, which constructs monotone sequences of lower and upper approximate solutions as well as leading to converge uniformly and monotonically to the unique solution of the given system, is attempted to describe for the functional differential system relative to both retardation and anticipation.

**AMS Subject Classification:** 65L05, 34A45
**Key Words:** systems with retardation and anticipation, monotone iterative technique

## 1. Introduction

Recently, in [2], for functional differential system with retardation and anticipation (see [3] and references therein),

$$\begin{cases} x'(t) = f(t, x(t), x_t, x^t), & t \in I = [t_0, T], \\ x_{t_0} = \phi_0, \quad x^T = \psi_0, & t_0 \geq 0, \ t_0 < T, \end{cases} \tag{1.1}$$

the monotone iterative technique has been applied successfully to obtain re-

Received: July 30, 3007 
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sults of existence and approximate of solutions, and the iterative schemes were developed by

$$\begin{cases} \alpha'_{n+1} = f(t, \alpha_n, \alpha_{n_t}, \beta_n^t) - M(\alpha_{n+1} - \alpha_n) - N\int_{-h_1}^{0}(\alpha_{(n+1)_t} - \alpha_{n_t})(s)ds, \\ \beta'_{n+1} = f(t, \beta_n, \beta_{n_t}, \alpha_n^t) - M(\beta_{n+1} - \beta_n) - N\int_{-h_1}^{0}(\beta_{(n+1)_t} - \beta_{n_t})(s)ds, \end{cases}$$

with $\alpha_{(n+1)_{t_0}} = \phi_0$, $\beta_{(n+1)_{t_0}} = \phi_0$, and, $\alpha_{n+1}^T$, $\beta_{n+1}^T$ chosen such that

$$\alpha_0^T \le \alpha_1^T \le \cdots \le \alpha_n^T \le \alpha_{n+1}^T \le \psi_0 \le \beta_{n+1}^T \le \beta_n \le \cdots \le \beta_1^T \le \beta_0^T. \tag{1.2}$$

An attempt here is, under upper and lower solution, to relax the limit condition for system (1.1), so that the monotone sequences of approximate solutions is ensured for the given system. Consequently, in this paper, we should revisit the system and follow up the idea developed in [2] to construct monotone sequences, by employing the iterative schemes

$$\begin{cases} \alpha'_{n+1} = & f(t, \alpha_n, \alpha_{n_t}, \alpha_n^t) - L(\alpha_{n+1} - \alpha_n) \\ & -N_1\int_{-h_1}^{0}(\alpha_{n_t} - \alpha_{(n+1)_t})(s)ds - N_2\int_0^{h_2}(\alpha_n^t - \alpha_{(n+1)}^t)(\sigma)d\sigma, \\ \beta'_{n+1} = & f(t, \beta_n, \beta_{n_t}, \beta_n^t) - L(\beta_{n+1} - \beta_n) \\ & -N_1\int_{-h_1}^{0}(\beta_{n_t} - \beta_{(n+1)_t})(s)ds - N_2\int_0^{h_2}(\beta_n^t - \beta_{(n+1)}^t)(\sigma)d\sigma, \end{cases}$$

with $\alpha_{(n+1)_{t_0}} = \phi_0$, $\alpha_{n+1}^T = \psi_0$ and $\beta_{(n+1)_{t_0}} = \phi_0$, $\beta_{n+1}^T = \psi_0$.

For a comprehensive introduction to the monotone iterative techniques we refer to [3].

## 2. Main Results

Consider the following functional differential system with retardation and anticipation

$$\begin{array}{ll} x'(t) = f(t, x(t), x_t, x^t), & t \in I = [t_0, T], \\ x_{t_0} = \phi_0, \quad x^T = \psi_0, & t_0 \ge 0, \; t_0 < T, \end{array} \tag{2.1}$$

where $\mathcal{C}_1 = C([-h_1, 0], R)$, $\mathcal{C}_2 = C([0, h_2], R)$, $\phi_0 \in \mathcal{C}_1$, $\psi_0 \in \mathcal{C}_2$ and $f \in C(I \times R \times \mathcal{C}_1 \times \mathcal{C}_2, R)$, $h_1$, $h_2 > 0$. Here $x_t = x_t(s) = x(t+s)$, $-h_1 \le s \le 0$, and $x^t = x^t(\sigma) = x(t+\sigma)$, $0 \le \sigma \le h_2$, represent retardation and anticipation, respectively.

In order to proceed further, we need the following known result relating to the linear functional differential inequalities from [2].

**Lemma 2.1.** *Suppose that $p \in C([t_0 - h_1, T + h_2], R)$, $p'(s)$ exists and is continuous on $I$ and*

$$p'(t) \le -Lp(t) + N_1 \int_{-h_1}^{0} p_t(s)ds + N_2 \int_0^{h_2} p^t(\sigma)d\sigma, \quad t \in I,$$

*where $L$, $N_1$, $N_2 > 0$, satisfying $N_1h_1 + N_2h_2 < L$. Then $p_{t_0} \le 0$, $p^T \le 0$ implies $p(t) \le 0$ on $I$.*

To discuss the monotone method for system (2.1), we need some assumptions, which we list below.

$(A_0)$ $\alpha_0$, $\beta_0 \in C^1(I, R)$, with $\alpha_0(t) \le \beta_0(t)$ on $I$, and satisfying

$$\begin{array}{l} \alpha_0'(t) \le f(t, \alpha_0(t), \alpha_{0_t}, \alpha_0^t), \quad \alpha_{0t_0} = \phi_1, \quad \alpha_0^T = \psi_1, \\ \beta_0'(t) \ge f(t, \beta_0(t), \beta_{0_t}, \beta_0^t), \quad \beta_{0t_0} = \phi_2, \quad \beta_0^T = \psi_2, \end{array}$$

such that $\phi_1 \le \phi_0 \le \phi_2$, $\psi_1 \le \psi_0 \le \psi_2$ on $I$ and $\phi_1$, $\phi_2 \in \mathcal{C}_1$, $\psi_1$, $\psi_2 \in \mathcal{C}_2$;

$(A_1)$ $f(t, x, \phi_1, \psi_1) - f(t, y, \phi_2, \psi_2) \ge -L(x - y) + N_1 \int_{-h_1}^{0} (\phi_1 - \phi_2)(s)ds + N_2 \int_0^{h_2} (\psi_1 - \psi_2)(\sigma)d\sigma$, for $\alpha_0(t) \le y \le x \le \beta_0(t)$, $\alpha_{0_t} \le \phi_2 \le \phi_1 \le \beta_{0_t}$, $\alpha_0^T \le \psi_2 \le \psi_1 \le \beta_0^T$, and $L$, $N_1$, $N_2 > 0$ with $N_1h_1 + N_2h_2 < L$.

**Theorem 2.1.** *Suppose that assumptions $(A_0)$-$(A_1)$ are satisfied. Then there exist monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ such that $\alpha_n(t) \to \rho(t)$, $\beta_n(t) \to \gamma(t)$ uniformly and monotonically as $n \to \infty$ on $[t_0 - h_1, T + h_2]$ and $(\rho, \gamma)$ are minimal and maximal solutions of system (2.1). Further*

*$(A_2)$ $f(t, x, \phi_1, \psi_1) - f(t, y, \phi_2, \psi_2) \le -\bar{L}(x - y) + \bar{N}_1 \int_{-h_1}^{0} (\phi_1 - \phi_2)(s)ds + \bar{N}_2 \int_0^{h_2} (\psi_1 - \psi_2)(\sigma)d\sigma$, for $\alpha_0(t) \le y \le x \le \beta_0(t)$, $\alpha_{0_t} \le \phi_2 \le \phi_1 \le \beta_{0_t}$, $\alpha_0^T \le \psi_2 \le \psi_1 \le \beta_0^T$, and $\bar{L}$, $\bar{N}_1$, $\bar{N}_2 > 0$ with $\bar{N}_1h_1 + \bar{N}_2h_2 < \bar{L}$, holds. Then $\rho(t) = \gamma(t) = x(t)$ is the unique solution of system (2.1) on $I$.*

*Proof.* Consider the following linear problem for each $n = 1, 2, 3, \dots$

$$\begin{array}{l} \alpha_{n+1}' = f(t, \alpha_n, \alpha_{n_t}, \alpha_n^t) - L(\alpha_{n+1} - \alpha_n) \\ -N_1 \int_{-h_1}^{0} (\alpha_{n_t} - \alpha_{(n+1)_t})(s)ds - N_2 \int_0^{h_2} (\alpha_n^t - \alpha_{(n+1)}^t)(\sigma)d\sigma, \\ \beta_{n+1}' = f(t, \beta_n, \beta_{n_t}, \beta_n^t) - L(\beta_{n+1} - \beta_n) \\ -N_1 \int_{-h_1}^{0} (\beta_{n_t} - \beta_{(n+1)_t})(s)ds - N_2 \int_0^{h_2} (\beta_n^t - \beta_{(n+1)}^t)(\sigma)d\sigma, \end{array} \tag{2.2}$$

with $\alpha_{(n+1)_{t_0}} = \phi_0$, $\alpha_{n+1}^T = \psi_0$ and $\beta_{(n+1)_{t_0}} = \phi_0$, $\beta_{n+1}^T = \psi_0$.

Clearly, each linear problem has a unique solution on $[t_0 - h_1, T + h_2]$. Our aim is to show that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_1 \le \beta_0 \quad \text{on} \quad I. \tag{2.3}$$

We first claim that $\alpha_0 \le \alpha_1$ on $I$. For this purpose, let $p = \alpha_0 - \alpha_1$, $t \in I$. Because of $(A_0)$ and (2.2), we get

$$\begin{aligned} p' &= \alpha_0' - \alpha_1' \le f(t,\alpha_0,\alpha_{0_t},\alpha_0^t) - f(t,\alpha_0,\alpha_{0_t},\alpha_0^t) \\ &\quad + L(\alpha_1 - \alpha_0) + N_1 \textstyle\int_{-h_1}^0 (\alpha_{0_t} - \alpha_{1_t})(s)ds + N_2 \int_0^{h_2} (\alpha_0^t - \alpha_1^t)(\sigma)d\sigma \\ &= -Lp + N_1 \textstyle\int_{-h_1}^0 p_t(s)ds + N_2 \int_0^{h_2} p^t(\sigma)d\sigma, \quad t \in I, \end{aligned}$$

and $p_{t_0} = \alpha_{0_{t_0}} - \alpha_{1_{t_0}} \le 0$, $p^T = \alpha_0^T - \alpha_1^T \le 0$.

By Lemma 2.1, this implies $p \le 0$ on $I$, and therefore we have $\alpha_0 \le \alpha_1$ on $I$. Similarly, we can show $\beta_1 \le \beta_0$ on $I$.

Next, we wish to prove that $\alpha_1 \le \beta_1$ on $I$. Consider $p = \alpha_1 - \beta_1$. In view of $(A_1)$, (2.2) and noting that $p_{t_0} = 0$, $p^T = 0$, we have

$$\begin{aligned} p' &= \alpha_1' - \beta_1' = f(t,\alpha_0,\alpha_{0_t},\alpha_0^t) - L(\alpha_1 - \alpha_0) \\ &\quad - N_1 \textstyle\int_{-h_1}^0 (\alpha_{0_t} - \alpha_{1_t})(s)ds - N_2 \int_0^{h_2} (\alpha_0^t - \alpha_1^t)(\sigma)d\sigma - f(t,\beta_0,\beta_{0_t},\beta_0^t) \\ &\quad + L(\beta_1 - \beta_0) + N_1 \textstyle\int_{-h_1}^0 (\beta_{0_t} - \beta_{1_t})(s)ds + N_2 \int_0^{h_2} (\beta_0^t - \beta_1^t)(\sigma)d\sigma \\ &\le -L(\alpha_0 - \beta_0) + N_1 \textstyle\int_{-h_1}^0 (\alpha_{0_t} - \beta_{0_t})(s)ds + N_2 \int_0^{h_2} (\alpha_0^t - \beta_0^t)(\sigma)d\sigma \\ &\quad - L(\alpha_1 - \alpha_0) - N_1 \textstyle\int_{-h_1}^0 (\alpha_{0_t} - \alpha_{1_t})(s)ds - N_2 \int_0^{h_2} (\alpha_0^t - \alpha_1^t)(\sigma)d\sigma \\ &\quad + L(\beta_1 - \beta_0) + N_1 \textstyle\int_{-h_1}^0 (\beta_{0_t} - \beta_{1_t})(s)ds + N_2 \int_0^{h_2} (\beta_0^t - \beta_1^t)(\sigma)d\sigma \\ &= -Lp + N_1 \textstyle\int_{-h_1}^0 p_t(s)ds + N_2 \int_0^{h_2} p^t(\sigma)d\sigma, \quad t \in I, \end{aligned}$$

which implies, using Lemma 2.1, $p \le 0$ on $I$ and consequently, $\alpha_1 \le \beta_1$ on $I$.

As a result, it follows that

$$\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0 \quad \text{on} \quad I. \tag{2.4}$$

Now suppose that for some $k > 1$, we get

$$\alpha_{k-1} \le \alpha_k \le \beta_k \le \beta_{k-1} \quad \text{on} \quad I. \tag{2.5}$$

Hence, we should show that

$$\alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k \quad \text{on} \quad I. \tag{2.6}$$

To do this, set $p = \alpha_k - \alpha_{k+1}$. Again, $p_{t_0} = 0$ and $p^T = 0$. By $(A_1)$ and (2.2), we see that

$$p' = \alpha_k' - \alpha_{k+1}' = f(t,\alpha_{k-1},\alpha_{(k-1)_t},\alpha_{k-1}^t) - L(\alpha_k - \alpha_{k-1})$$
$$- N_1 \int_{-h_1}^0 (\alpha_{(k-1)_t} - \alpha_{k_t})(s)ds - N_2 \int_0^{h_2} (\alpha_{k-1}^t - \alpha_k^t)(\sigma)d\sigma - f(t,\alpha_k,\alpha_{k_t},\alpha_k^t)$$

$$
\begin{aligned}
&+ L(\alpha_{k+1}-\alpha_k) + N_1\int_{-h_1}^{0}(\alpha_{k_t}-\alpha_{(k+1)_t})(s)ds + N_2\int_0^{h_2}(\alpha_k^t-\alpha_{k+1}^t)(\sigma)d\sigma \\
&\le -L(\alpha_{k-1}-\alpha_k) + N_1\int_{-h_1}^{0}(\alpha_{(k-1)_t}-\alpha_{k_t})(s)ds + N_2\int_0^{h_2}(\alpha_{k-1}^t-\alpha_k^t)(\sigma)d\sigma \\
&- L(\alpha_k-\alpha_{k-1}) - N_1\int_{-h_1}^{0}(\alpha_{(k-1)_t}-\alpha_{k_t})(s)ds - N_2\int_0^{h_2}(\alpha_{k-1}^t-\alpha_k^t)(\sigma)d\sigma \\
&+ L(\alpha_{k+1}-\alpha_k) + N_1\int_{-h_1}^{0}(\alpha_{k_t}-\alpha_{(k+1)_t})(s)ds + N_2\int_0^{h_2}(\alpha_k^t-\alpha_{k+1}^t)(\sigma)d\sigma \\
&\qquad\qquad = -Lp + N_1\int_{-h_1}^{0}p_t(s)ds + N_2\int_0^{h_2}p^t(\sigma)d\sigma, \quad t\in I,
\end{aligned}
$$

which, applying Lemma 2.1, yields $\alpha_k \le \alpha_{k+1}$ on $I$. A similar argument holds for $\beta_{k+1} \le \beta_k$ on $I$.

To prove $\alpha_{k+1} \le \beta_{k+1}$ on $I$. Taking $p = \alpha_{k+1} - \beta_{k+1}$ so that $p_{t_0} = 0$, $p^T = 0$, and proceeding as before, one can obtain that

$$
p' \le -Lp + N_1\int_{-h_1}^{0}p_t(s)ds + N_2\int_0^{h_2}p^t(\sigma)d\sigma, \quad t\in I,
$$

which follows that $\alpha_{k+1} \le \beta_{k+1}$ on $I$. And, by induction, (2.3) is valid on $I$.

Further, since the sequences $\{\alpha_n\}$, $\{\beta_n\}$ are bounded by (2.3), employing the standard arguments [1], one can conclude that $\{\alpha_n\}$, $\{\beta_n\}$ converge uniformly on $I$, that is, $\alpha_n \to \rho$, $\beta_n \to \gamma$ uniformly on $I$. Also, it is now easy to show that $(\rho,\gamma)$ satisfy

$$
\begin{aligned}
\rho' &= f(t,\rho,\rho_t,\rho^t), \quad \rho_{t_0} = \phi_0, \quad \rho^T = \psi_0, \\
\gamma' &= f(t,\gamma,\gamma_t,\gamma^t), \quad \gamma_{t_0} = \phi_0, \quad \gamma^T = \psi_0,
\end{aligned}
$$

with $\rho \le \gamma$.

Now we claim that $(\rho,\gamma)$ are minimal and maximal solutions of system (2.1). Define $x(t)$ be any solution of system (2.1) with $x_{t_0} = \phi_0$, $x^T = \psi_0$ such that $\alpha_0 \le x \le \beta_0$ on $I$. Then it is enough to get that $\rho \le x \le \gamma$. Put $p = \alpha_1 - x$, note that $p_{t_0} = 0$, $p^T = 0$. We arrive at

$$
\begin{aligned}
p' &= \alpha_1' - x' = f(t,\alpha_0,\alpha_{0_t},\alpha_0^t) - L(\alpha_1-\alpha_0) \\
&\quad - N_1\textstyle\int_{-h_1}^{0}(\alpha_{0_t}-\alpha_{1_t})(s)ds - N_2\int_0^{h_2}(\alpha_0^t-\alpha_1^t)(\sigma)d\sigma - f(t,x,x_t,x^t) \\
&\le -Lp + N_1\textstyle\int_{-h_1}^{0}p_t(s)ds + N_2\int_0^{h_2}p^t(\sigma)d\sigma, \quad t\in I,
\end{aligned}
$$

Thus, from Lemma 2.1, we have $\alpha_1 \le x$ on $I$. Similarly, $x \le \beta_1$ on $I$. Arguing as before, it follows that $\alpha_{n+1} \le x \le \beta_{n+1}$ on $I$. Hence, $(\rho,\gamma)$ are minimal and maximal solutions of system (2.1).

Finally, if, in addition, condition $(A_2)$ holds, since $\rho \leq \gamma$, we consider $p = \gamma - \rho$ and again $p_{t_0} = 0$, $p^T = 0$. Then in view of $(A_2)$, we obtain

$$\begin{aligned} p' &= \gamma' - \rho' = f(t, \gamma, \gamma_t, \gamma^t) - f(t, \rho, \rho_t, \rho^t) \\ &\leq -\bar{L}p + \bar{N}_1 \int_{-h_1}^{0} p_t(s)ds + \bar{N}_2 \int_0^{h_2} p^t(\sigma)d\sigma, \quad t \in I, \end{aligned}$$

which implies by Lemma 2.1, $p(t) \leq 0$ on $I$ and therefore $\rho = \gamma$ on $I$. Thus $x = \rho = \gamma$ is the unique solution of system (2.1) with $x_{t_0} = \phi_0$, $x^T = \psi_0$. □

## Acknowledgements

Supported by the Key Project of Chinese Ministry of Education (207014) and the Natural Science Foundation of Hebei Province of P.R. China (A20060009 41).

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