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BANACH SPACE VALUED SEQUENCE SPACE $l_M(X,p)$

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Abstract: In this paper, we introduce the Banach space valued sequence space $l_M(X,p)$ and examine various algebraic and topological properties of this space. Finally we introduce and investigate some topological properties of a subspace of $l_M(X,p)$.

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1. Introduction

Lindenstrauss and Tzafriri [5] used the idea of an Orlicz function M to construct the sequence space l_M of all sequences of scalars (x_k) such that $\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty$ for some $\rho > 0$. The space l_M equipped with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}$$

is a BK space [2, p. 300] usually called an Orlicz sequence space. The space l_M is closely related to the space l_p which is an Orlicz sequence space with $M(x) = x^p$, $1 \leq p < \infty$. We recall [2, 5] that an Orlicz function M is a function from $[0,\infty)$ to $[0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for all $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Note that an Orlicz function is always unbounded.

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An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$. It is easy to see [3] that always $K > 2$. A simple example of an Orlicz function which satisfies the Δ_2-condition for all values of u is given by $M(u) = a|u|^\alpha (\alpha > 1)$, since $M(2u) = a2^\alpha |u|^\alpha = 2^\alpha M(u)$. The Orlicz function $M(u) = e^{|u|} - |u| - 1$ does not satisfy the Δ_2-condition.

The Δ_2-condition is equivalent to the inequality $M(lu) \leq K(l)M(u)$ which holds for all values of u, where l can be any number greater than unity.

An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x p(t)dt\,,$$

where p known as the kernel of M, is right differentiable for $t \geq 0$, $p(0) = 0, p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

Before proceeding with the main results we recall [6] some terminology and notations.

A paranormed space $X = (X, g)$ is a topological linear space in which the topology is given by a paranorm g; a real subadditive function on X such that $g(\theta) = 0, g(x) = g(-x)$ and such that the scalar multiplication is continuous. In the above, θ is the zero in the complex linear space X and continuity of multiplication means that $\lambda_n \to \lambda, x_n \to x$ (i.e., $g(x_n - x) \to 0$) imply $\lambda_n x_n \to \lambda x$ (i.e., $g(\lambda_n x_n - \lambda x) \to 0$), for scalars λ and vectors x.

A paranorm for which $g(x) = 0$ implies $x = \theta$ is called total paranorm.

A Fréchet space is a complete metric linear space, or equivalently a complete totally paranormed space.

Let w denote the space of all complex sequences $x = (x_k)$. Let X be a linear subspace of w such that X is a Fréchet space with continuous coordinate projections. Then we say that X is an FK space, or a Fréchet coordinat space. If the metric of an FK space X is given by a complete norm then we say that X is a BK space, i.e., a Banach coordinat space.

A sequence (b_k) of elements of a paranormed space (X, g) is called a Schauder basis for X if and only if, for each $x \in X$, there exists a unique sequence (λ_k) of scalars such that $x = \sum_{k=1}^\infty \lambda_k b_k$, i.e., such that $g(x - \sum_{k=1}^n \lambda_k b_k) \to 0\,(n \to \infty)$.

An FK space X has AK, or has the AK property, if (e_k), the sequence of unit vectors, is a Schauder basis for X. In effect, this means that for each $x = (x_k) \in X$ we have $(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \sum_{k=1}^n x_k e_k \to x\,(n \to \infty)$, where the convergence is in the metric of X.

Let $(X, \|\,.\,\|)$ be a Banach space over the complex field $\mathbb{C}$. Denote by $w(X)$ the space of all X-valued sequences. Let M be an Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers.

We now introduce the Banach space valued sequence space $l_M(X,p)$ using an Orlicz function M as follows:

$$l_M(X,p) = \left\{ x \in w(X) : \sum_{k=1}^{\infty} \left[M\left(\frac{\|x_k\|}{\rho} \right) \right]^{p_k} < \infty \, \text{for some} \, \rho > 0 \right\}.$$

Some well-known spaces are obtained by specializing X, M and p.

(i) If $X = \mathbb{C}, p_k = 1$ for all k, then $l_M(X,p) = l_M$ (Lindenstrauss and Tzafriri [5]).

(ii) If $X = \mathbb{C}$, then $l_M(X,p) = l_M(p)$ (Parashar and Choudhary [7]).

(iii) If $M(x) = x$ and $p_k = p(1 \le p < \infty)$ for all k, then $l_M(X,p) = l_p(X)$ (Leonard [4]).

In Section 2, we propose to study various algebraic and topological properties of the sequence space $l_M(X,p)$. In Section 3, certain inclusion relations between $l_M(X,p)$ spaces have been established. In Section 4, some information on multipliers for $l_M(X,p)$ is given. In Section 5, a subspace of $l_M(X,p)$ has been introduced and some topological properties of it have been discussed.

The following inequalities (see, e.g., [6; p. 190]) are needed throughout the paper.

Let $p = (p_k)$ be a bounded sequence of positive real numbers. If $H = \sup_k p_k$, then for any complex a_k and b_k ,

$$|a_k + b_k|^{p_k} \le C(|a_k|^{p_k} + |b_k|^{p_k}) , \tag{1.1}$$

where $C = \max(1, 2^{H-1})$. Also for any complex λ,

$$|\lambda|^{p_k} \le \max(1, |\lambda|^H) . \tag{1.2}$$

2. Linear Topological Structure of $l_M(X,p)$ Spaces

Theorem 2.1. *For any Orlicz function M, $l_M(X,p)$ is a linear space over the complex field $\mathbb{C}$.*

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.2. *$l_M(X,p)$ is a topological linear space, paranormed by*

$$g(x) = \inf\left\{\rho^{p_n/G} : \left(\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k}\right)^{\frac{1}{G}} \leq 1\right\}, \tag{2.1}$$

where $G = \max(1, \sup_k p_k)$.

The proof uses ideas similar to those used (e.g.) in [7, p. 421] and the fact that every paranormed space is a topological linear space [8, p. 37].

Corollary 2.3. *If p is a constant sequence, then $l_M(X,p)$ is a normed space for $p \geq 1$ and a p-normed space for $p < 1$.*

Definition 2.4. (see [1]) A linear subspace Y of $w(X)$ is a generalized FK space (resp. a generalized BK space) if Y is a Fréchet space (resp. a Banach space) with continuous coordinate projections.

In case $X = \mathbb{C}$, then Y becomes an FK space (resp. a BK space).

Proposition 2.5. (see [2, p. 300]) *We have, for x in l_M, the inequality*

$$\sum_{i\geq 1} M\left(\frac{|x_i|}{\|x\|_{(M)}}\right) \leq 1,$$

where $\|x\|_{(M)} = \inf\left\{k > 0 : \sum_{i\geq 1} M\left(\frac{|x_i|}{k}\right) \leq 1\right\}$.

Theorem 2.6. *Let $1 \leq p_k < \infty$, then $l_M(X,p)$ is a generalized FK space paranormed by (2.1).*

Proof. Let (x^i) be a Cauchy sequence in $l_M(X,p)$. Let r and x_0 be fixed. Then for each $\frac{\epsilon}{rx_0} > 0$ there exists a positive integer N such that

$$g(x^i - x^j) < \frac{\epsilon}{rx_0}, \text{ for all } i, j \geq N.$$

Using (2.1) and Proposition 2.5, we get $\left(\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k^i - x_k^j\|}{g(x^i - x^j)}\right)\right]^{p_k}\right)^{1/G} \leq 1.$ Thus

$$\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k^i - x_k^j\|}{g(x^i - x^j)}\right)\right]^{p_k} \leq 1.$$

Since $1 \leq p_k < \infty$, it follows that $M\left(\frac{\|x_k^i - x_k^j\|}{g(x^i - x^j)}\right) \leq 1$, for each $k \geq 1$.

We choose $r > 0$ such that $(\frac{x_0}{2})rp(\frac{x_0}{2}) \geq 1$, where p is the kernel associated with M. Hence,

$$M\left(\frac{\|x_k^i - x_k^j\|}{g(x^i - x^j)}\right) \le \left(\frac{x_0}{2}\right) rp\left(\frac{x_0}{2}\right)$$

for each $k \in \mathbb{N}$. Using the integral representation of Orlicz function M, we get

$$\|x_k^i - x_k^j\| \le \frac{rx_0}{2} g(x^i - x^j) < \frac{\epsilon}{2}\ , \ \text{for all}\, i,j \ge N.$$

Hence for each k, (x_k^i) is a Cauchy sequence in X, whence by the completeness of X there exists a sequence $x = (x_k)$ such that $x_k \in X$ for each $k \in \mathbb{N}$ and $\lim_{i\to\infty} x_k^i = x_k$ for each k. For given $\epsilon > 0$, choose an integer $n_0 > 1$ such that $g(x^i - x^j) < \epsilon$ for all $i, j \ge n_0$ and a $\rho > 0$, such that $g(x^i - x^j) < \rho < \epsilon$. Since

$$\left(\sum_{k=1}^{n}\left[M\left(\frac{\|(x_k^i - x_k^j)\|}{\rho}\right)\right]^{p_k}\right)^{1/G} \le 1, \text{for all}\, i,j \ge n_0.$$

Now, using continuity of M and taking $j \to \infty$ in the above inequality, we get

$$\left(\sum_{k=1}^{n}\left[M\left(\frac{\|(x_k^i - x_k)\|}{\rho}\right)\right]^{p_k}\right)^{1/G} \le 1, \text{for all}\, i \ge n_0.$$

Letting $n \to \infty$, we get $g(x^i - x) < \rho < \epsilon$ for all $i \ge n_0$. Thus (x^i) converges to x in the paranorm of $l_M(X,p)$. Since $(x^i) \in l_M(X,p)$ and M is continuous, it follows that $x \in l_M(X,p)$.

We next show that the coordinate functionals $P_i : l_M(X,p) \to X$, where $P_i(x) = x_i$ are continuous.

For $\epsilon > 0$ let $\delta > 0$ be such that $0 < \delta < 1$ and $\delta \le M(\epsilon)$. Let $g(x) < \delta$ so that $\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k\|}{g(x)}\right)\right]^{p_k} \le 1$. This implies that $\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k\|}{\delta}\right)\right]^{p_k} \le 1$, and so

$$\left[M\left(\frac{\|x_k\|}{\delta}\right)\right]^{p_k} \le 1 \quad \text{for each}\, k \ge 1.$$

As $1 \le p_k < \infty$, so $M\left(\frac{\|x_k\|}{\delta}\right) \le 1$ for each $k \ge 1$.

Since $0 < \delta < 1$ and M is convex, $\frac{1}{\delta}M(\|x_k\|) \le M\left(\frac{\|x_k\|}{\delta}\right) \le 1$ which implies that $M(\|x_k\|) \le \delta \le M(\epsilon)$. Since M is non-decreasing, we have $\|x_k\| < \epsilon$ for each $k \ge 1$. Thus the coordinate functionals are continuous and this completes the proof of the theorem.

Corollary 2.7. *If p is a constant sequence and $p \ge 1$, then $l_M(X,p)$ is a generalized BK space.*

3. Inclusion Between $l_M(X,p)$ Spaces

We now investigate some inclusion relations between $l_M(X,p)$ spaces.

Theorem 3.1. *If $p=(p_k)$ and $q=(q_k)$ are bounded sequences of positive real numbers with $0<p_k\le q_k<\infty$ for each k, then for any Orlicz function M, $l_M(X,p)\subseteq l_M(X,q)$.*

Proof. Let $x\in l_M(X,p)$. Then there exists some $\rho>0$ such that $\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k}<\infty$. This implies that $M\left(\frac{\|x_k\|}{\rho}\right)\le 1$ for sufficiently large values of k, say $k\ge n_0$ for some fixed $n_0\in\mathbb{N}$. Since M is non-decreasing and $p_k\le q_k$, we have

$$\sum_{k\ge n_0}^{\infty}\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{q_k}\le\sum_{k\ge n_0}^{\infty}\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k}<\infty.$$

This shows that $x\in l_M(X,q)$ and completes the proof. □

Theorem 3.2. *If $r=(r_k)$ and $t=(t_k)$ are bounded sequences of positive real numbers with $0<r_k,t_k<\infty$ and if $p_k=\min(r_k,t_k)$, $q_k=\max(r_k,t_k)$, then for any Orlicz function M, $l_M(X,p)=l_M(X,r)\bigcap l_M(X,t)$ and $l_M(X,q)=G$, where G is the subspace of w generated by $l_M(X,r)\bigcup l_M(X,t)$.*

Proof. It follows from Theorem 3.1 that $l_M(X,p)\subseteq l_M(X,r)\bigcap l_M(X,t)$ and that $G\subseteq l_M(X,q)$.

For any complex λ, $|\lambda|^{p_k}\le\max(|\lambda|^{r_k},|\lambda|^{t_k})$; thus $l_M(X,r)\bigcap l_M(X,t)\subseteq l_M(X,p)$.

Let $A=\{k:r_k\ge t_k\}$ and $B=\{k:r_k<t_k\}$. If $x=(x_k)\in l_M(X,q)$, we write

$$y_k=x_k(k\in A)\,,\quad y_k=0(k\in B)\,,\quad z_k=0(k\in A)\,,\quad\text{and}\quad z_k=x_k(k\in B).$$

Then since $x=(x_k)\in l_M(X,q)$, there exists some $\rho>0$ such that $\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{q_k}<\infty$. Now,

$$\sum_{k=1}^{\infty}\left[M\left(\frac{\|y_k\|}{\rho}\right)\right]^{r_k}=\sum_{k\in A}+\sum_{k\in B}=\sum_{k\in A}\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{q_k}<\infty$$

and so $y\in l_M(X,r)\subseteq G$. Similarly, $z\in l_M(X,t)\subseteq G$. Thus, $x=y+z\in G$. We have proved that $l_M(X,q)\subseteq G$, which gives the required result. □

Corollary 3.3. *The three conditions $l_M(X,r)\subseteq l_M(X,t)$, $l_M(X,p)=l_M(X,r)$ and $l_M(X,t)=l_M(X,q)$ are equivalent.*

Corollary 3.4. *$l_M(X,r)=l_M(X,t)$ if and only if $l_M(X,p)=l_M(X,q)$.*

4. The Space of Multipliers of $l_M(X,p)$

For any set $E \subset w(X)$ the space of multipliers of E, denoted by $S(E)$, is given by

$$S(E) = \{a = (a_k) \in w(X) : ax = (a_k x_k) \in E \text{ for all } x = (x_k) \in E\}.$$

Theorem 4.1. *For an Orlicz function M which satisfies the Δ_2-condition and Banach algebra X, we have $l_\infty(X) \subseteq S[l_M(X,p)]$, where $l_\infty(X) = \{a = (a_k) \in w(X) : \sup_k \|a_k\| < \infty\}$.*

Proof. Let $a = (a_k) \in l_\infty(X)$, $T = \sup_k \|a_k\|$ and $x = (x_k) \in l_M(X,p)$. Then $\sum_{k=1}^{\infty} \left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k} < \infty$ for some $\rho > 0$. Since M satisfies the Δ_2-condition, there exists a constant K such that

$$\sum_{k=1}^{\infty} \left[M\left(\frac{\|a_k x_k\|}{\rho}\right)\right]^{p_k} \le \sum_{k=1}^{\infty} \left[M\left(\frac{\|a_k\|\|x_k\|}{\rho}\right)\right]^{p_k}$$

$$\le \sum_{k=1}^{\infty} \left[M\left((1+[T])\frac{\|x_k\|}{\rho}\right)\right]^{p_k} \le (K(1+[T]))^H \sum_{k=1}^{\infty} \left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k} < \infty,$$

where $[T]$ denotes the integer part of T. Hence $a \in S[l_M(X,p)]$.

5. A Subspace of $l_M(X,p)$

In this section we introduce a subspace of $l_M(X,p)$ and investigate some topological properties of it.

We define $h_M(X,p)$ by

$$h_M(X,p) = \left\{x = (x_k) \in w(X) : \sum_{k=1}^{\infty} \left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k} < \infty \text{ for every } \rho > 0\right\}.$$

The space $h_M(X,p)$ is clearly a subspace of $l_M(X,p)$, and the topology is determined by the paranorm of $l_M(X,p)$ given by (2.1).

Theorem 5.1. *Let $1 \le p_k < \infty$. Then $h_M(X,p)$ is a generalized FK space with the paranorm given by (2.1).*

Proof. Since $h_M(X,p)$ is a subspace of $l_M(X,p)$ and $l_M(X,p)$ is a generalized FK space under the paranorm given by (2.1), so it is sufficient to show that $h_M(X,p)$ is closed in $l_M(X,p)$. Therefore, let $(x^i) = ((x_k^i))$ be a sequence in $h_M(X,p)$ such that $g(x^i - x) \to 0$ as $i \to \infty$, where $x = (x_k) \in l_M(X,p)$.

To complete the proof we need show that $\sum_{k=1}^{\infty} \left[M\left(\frac{\|x_k\|}{\xi}\right)\right]^{p_k} < \infty$ for every

$\xi > 0$. To $\xi > 0$ there corresponds an integer m such that $g(x^m - x) < \xi/2$. Since M is non-decreasing and convex,

$$\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k\|}{\xi}\right)\right]^{p_k} = \sum_{k=1}^{\infty}\left[M\left(\frac{2\|x_k^m\| + 2(\|x_k\| - \|x_k^m\|)}{2\xi}\right)\right]^{p_k}$$
$$\leq \sum_{k=1}^{\infty}\frac{1}{2^{p_k}}\left[M\left(\frac{\|x_k^m\|}{\xi/2}\right) + M\left(\frac{\|x_k - x_k^m\|}{\xi/2}\right)\right]^{p_k} < \sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k^m\|}{\xi/2}\right)\right.$$
$$\left.+M\left(\frac{\|x_k^m - x_k\|}{\xi/2}\right)\right]^{p_k} \leq C\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k^m\|}{\xi/2}\right)\right]^{p_k} + C\sum_{k=1}^{\infty}\left[M\left(\frac{\|x_k^m - x_k\|}{g(x^m - x)}\right)\right]^{p_k}$$
$< \infty$, by Proposition 2.5 ,

where $C = \max(1, 2^{H-1})$. Thus $x \in h_M(X,p)$ which shows that $h_M(X,p)$ is complete.

Let X be any Banach algebra with identity e. Then for each $k \in \mathbb{N}$ we denote by $e_k(X) = (\theta, \theta, \ldots, e, \theta, \theta, \ldots)$ the k-th unit vector in $w(X)$, so that $e_k(X)$ is the sequence with e (identity of Banach algebra) in k-th place and θ elsewhere.

Definition 5.2. (see [1]) A generalized FK space $Y \subset w(X)$ is a generalized AK space if $(e_k(X))$, the sequence of unit vectors in $w(X)$, is a Schauder basis for X, i.e., for each $y = (y_k) \in Y$ we have $(y_1, y_2, \ldots, y_n, \theta, \theta, \ldots) = \sum_{k=1}^{n} y_k e_k(X) \to y (n \to \infty)$, where the convergence is in the metric of Y.

In case $X = \mathbb{C}$, then Y becomes an AK space.

Proposition 5.3. *If X is a Banach algebra with identity and $1 \leq p_k < \infty$, then $h_M(X,p)$ is a generalized AK space.*

Proof. Let $x = (x_k) \in h_M(X,p)$. For each ϵ, $0 < \epsilon < 1$, we can find an arbitrary positive integer n_0 such that $\sum_{k \geq n_0}\left[M\left(\frac{\|x_k\|}{\epsilon}\right)\right]^{p_k} \leq 1$. Write $y_n = x - (x_1, x_2, \ldots, x_n, \theta, \theta, \ldots)$, so that $y_n = x - \sum_{k=1}^{n} x_k e_k(X) = (\theta, \theta, \ldots, \theta, x_{n+1}, x_{n+2}, \ldots)$ and hence for $n \geq n_0$

$$g(y_n) = \inf\left\{\rho^{p_n/G} > 0 : \left(\sum_{k \geq n+1}\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k}\right)^{1/G} \leq 1\right\}$$
$$\leq \inf\left\{\rho^{p_n/G} > 0 : \left(\sum_{k \geq n}\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k}\right)^{1/G} \leq 1\right\} < \epsilon,$$

which implies $x = \sum_{k=1}^{\infty} x_k e_k(X)$.

This representation for x is unique, for if we also had $x = \sum_{k=1}^{\infty} \lambda_k e_k(X)$ then, as $n \to \infty$,

$$g\left(\sum_{k=1}^{n} (\lambda_k - x_k) e_k(X)\right) \le g\left(x - \sum_{k=1}^{n} x_k e_k(X)\right) + g\left(x - \sum_{k=1}^{n} \lambda_k e_k(X)\right) \to 0,$$

whence, $|\lambda_1 - x_1| + |\lambda_2 - x_2| + \ldots + |\lambda_n - x_n| \to 0$, which implies $\lambda_k = x_k$ for all $k = 1, 2, \ldots$ and this completes the proof of the theorem. □

We now introduce the following definition:

Definition 5.4. (see [2]) An Orlicz function M is said to satisfy the Δ_2-condition for small x or at 0 if for each $k > 0$ there exist $R_k > 0$ and $x_k > 0$ such that $M(kx) \le R_k M(x)$ for all $x \in (0, x_k]$.

Proposition 5.5. *Let X be a Banach algebra with identity, $1 \le p_k < \infty$ and M be an Orlicz function which satisfies the Δ_2-condition at 0, then $l_M(X,p)$ is a generalized AK space.*

Proof. In view of Proposition 5.3 it is sufficient to show that $l_M(X,p) \subseteq h_M(X,p)$. Let $x \in l_M(X,p)$, then for $\rho > 0$, $\sum_{k=1}^{\infty} \left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k} < \infty$ and this implies $\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k} \to 0$ as $k \to \infty$, and hence

$$M\left(\frac{\|x_k\|}{\rho}\right) \to 0, \quad \text{as} \quad k \to \infty. \tag{5.1}$$

Choose an arbitrary $\eta > 0$. If $\rho \le \eta$, then $\sum_k \left[M\left(\frac{\|x_k\|}{\eta}\right)\right]^{p_k} < \infty$. Let now $\eta < \rho$ and put $\xi = \rho/\eta$. Since M satisfies the Δ_2-condition at 0 so for each $\xi > 0$ there exists $R \equiv R_\xi > 0$ and $r \equiv r_\xi > 0$ with $M(\xi x) \le RM(x)$ for all x in $(0, r]$.

By (5.1) there exists an I in $\mathbb{N}$ such that $M\left(\frac{\|x_k\|}{\rho}\right) < \frac{1}{2}\, rp\left(\frac{r}{2}\right)$, for all $k \ge I$, and the last inequality yields

$$\frac{\|x_k\|}{\rho} \le r \quad \text{forall} \quad k \ge I. \tag{5.2}$$

For otherwise, we can find $j > I$ with $\frac{\|x_k\|}{\rho} > r$, and thus

$$M\left(\frac{\|x_j\|}{\rho}\right) \geq \int_{r/2}^{\frac{\|x_j\|}{\rho}} p(t)dt > \frac{1}{2}\, rp\left(\frac{r}{2}\right)$$

This contradiction establishes (5.2). Using (5.2), we get

$$\sum_{k\geq I}\left[M\left(\frac{\|x_k\|}{\eta}\right)\right]^{p_k} = \sum_{k\geq I}\left[M\left(\frac{\xi\|x_k\|}{\rho}\right)\right]^{p_k} \leq \sum_{k\geq I}\left[RM\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k}$$

$$\leq \max(1, R^H)\sum_{k\geq I}\left[M\left(\frac{\|x_k\|}{\rho}\right)\right]^{p_k} < \infty\,,$$

and hence $\sum_{k\geq 1}\left[M\left(\frac{\|x_k\|}{\eta}\right)\right]^{p_k} < \infty$ for every $\eta > 0$. Thus $x \in h_M(X,p)$. This completes the proof. □

Corollary 5.6. *If X is a Banach algebra with identity, then for an Orlicz function M which satisfies the Δ_2-condition at 0 and a constant sequence p, $l_M(X,p)$ is a generalized AK-BK space.*

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