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ON TOPOLOGICAL DIMENSION AND DIGITAL TOPOLOGY

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Abstract: An algorithm for computing topological dimension is suggested and explained. Illustrative examples for the computation process are included. Semi open sets in digital topology are constructed. The concept of semi boundary of subsets in topological spaces is used in introducing a new type of dimensions, called semi inductive dimension. The digital topology on the set of integers is proved as an example for a topology whose semi inductive dimension is different from its corresponding classical inductive dimension. Some properties and characterizations of the introduced concept are obtained.

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1. Introduction

The concept of topological space can be described as a generalization of geometrical shapes and for the grand universe we live in. The abstractness of topological space and its extended views opened the way for new fields of applications for topological space theories, for example: theory of computation and programming [9], decision rule extraction [13], feature selection [13, 16], granular computing [17], biochemistry [15], image analysis [10], etc.

It is known from various lines of attack (intuitionistic and constructive mathematics, recursion theory, domain theory, programming language semantics)

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that domains of computation, or data types, are topological spaces, and that computable maps are continuous. Moreover, the topologies that arise are familiar. For instance, computable functions on infinite sequences of binary digits are continuous with respect to the Cantor topology. Many applications of the topology of data types are known in the theory of computation [9].

Dimension of topological space gives a precise parameterization of the conceptual or visual complexity of the space and its subsets. In fact, the concept can even be applied to abstract objects which cannot be directly visualized.

For example, the notion of time can be considered as one-dimensional, since it can be thought of as consisting of only “now”, “before” and “after”. Since “before” and “after” regardless of how far back or how far into the future they are, are extensions, time is like a line, a one-dimensional object.

There are several branches and extensions of the notion of topological dimension. Implicit in the notion of the Lebesgue covering dimension is that dimension, in a sense, is a measure of how an object fills space. If it takes up a lot of room, it is higher dimensional, and if it takes up less room, it is lower dimensional. Hausdorff dimension (also called fractal dimension) is a fine tuning of this definition that allows notions of objects with dimensions other than integers. Fractals are objects whose Hausdorff dimension is different from their topological dimension.

The measurements of any physical object depend basically on the dimension of the space to which the object is related. However, the determination of the Euclidean dimension is well known. But for topological spaces, which are not Euclidean, the computation of dimensions did not take the suitable interest of topologists for a long time. To the best of our knowledge, most of the studies about topological dimension are concentrated on theoretical cases, for example see [1, 5]. The computation for dimensions of abstract spaces have not been discussed.

Recently, the developments in computer science and the importance of topological applications in high energy physics have directed attention to the importance of topological dimensions not only in pure mathematics but also in many fields of applications. For example the computation of topological dimensions for discrete samples [4] and the application of fractal dimension in feature selection [16], in high energy physics and in Elnaschie theories [6, 7, 8]. In fact, talks of Professor Elnaschie and his works are the main motivation to author interest in topological dimension

The purpose of this paper is to study two basic cases of topological dimensions and give examples for computing these types. Moreover, we use the class of semiopen sets to introduce a method for computing dimensions and study some of its properties and characterizations.

In Section 2 we explain in detail the concept of inductive dimension, we give some examples for obtaining this dimension. The main goal of Section 3 is to introduce a type of inductive dimensions called semi-inductive dimension using topological concepts based on semiopen sets and investigate some of its properties. Also we give examples for this type of dimensions.

2. Inductive Dimensions

In view of Euclidean topology, points have dimension zero and the curve is one dimensional, where the walls (the boundary) of the curve are discrete points which are a zero dimensional, and the surface is two dimensional, where its walls are lines, which have dimension one, the solid body is three dimensional where the walls of the room are surfaces which are two dimensional. From the above discussion, we notice that all dimensions are dependent on the concept of boundary. We aim in this article to study the inductive dimension as a type of topological dimensions whose computation is a slight generalization of Euclidean dimensions.

Definition 1. (see [1]) Let (X, τ) be a topological space, then (X, τ) has inductive dimension equal to (-1) if and only if $X = \phi$ which is denoted by $indX = -1$.

Remark 1. The above discussion and Definition 2.1 led to our belief that $ind\phi = -1$ is a result of postulating that each point has a dimension zero.

Definition 2. (see [1]) Let n be an integer larger than or equal zero, then (X, τ) has inductive dimension less than or equal n, if it has a base β such that for every $B \in \beta$ **,** the boundary Fr(B) has inductive dimension less than or equal (n-1), and we write $indX \leq n$.

Definition 3. (see [1]) Let (X, τ) be topological space. If X has inductive dimension less than or equal n, and if it is false that X has inductive dimension less than or equal (n-1), then the inductive dimension of X is n, and denoted by $indX = n$.

The following is the definition of the infinite inductive dimension.

Definition 4. (see [1]) If for every $n \in Z_+$ it is false that X has inductive dimension less than or equal n, then X is said to have an infinite inductive dimension, and we write *ind* X $= \infty$.

Thus we conclude that the dimension of any set increased by one over the dimension of its boundary. According to this conclusion, we introduce the following algorithm for computing dimension.

Remark 2. To compute the inductive dimension of the topology of the space (X, τ), we take a member B of the base of τ and find the boundary Fr B.

(1) If Fr B is ϕ then the inductive dimension of Fr B is - 1.

(2) If Fr B is not ϕ, we consider Fr B as a subspace and compute the boundary of all members of the base of τ_{FrB}.

(3) Repeat the above steps until the boundary is ϕ for all members of the base.

(4) If n is the number of steps to arrive that all members B of the base has an empty boundary, then the inductive dimension of the space is less than or equal (n-1), then $indX \leq n - 1$.

We introduce the following examples to indicate the above steps.

Example 1. Let $X = \{a, b, c, d\}$, $\beta = \{\{a\}, \{a, b\}, \{a, c, d\}\}$ and

$$\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}\}.$$

After two steps Fr (B) $= \phi$ for every B $\in \beta$,

$$indX = 2 + (-1) = 1.$$

Then $indX = 1$.

Example 2. (see [1]) Let X be the Sierpinski space X= {0, 1} and $\tau = \{\phi, \{0, 1\}, \{0\}\}$.

Then $indX \leq 1$.

Example 3. Let $X = \{a, b, c, d, e\}$, $\beta = \{\{a, b, c\}, \{a, c, e\}, \{c, d\}, \{a, c\}, \{c\}\}$ and

$$\tau = \left\{ \begin{array}{l} \phi, X, \{a, b, c\}, \{a, c, e\}, \{c, d\}, \{a, c\}, \{c\}, \{a, b, c, e\}, \\ \{a, b, c, d\}, \{a, c, e, d\}, \{a, c, d\} \end{array} \right\}.$$

Here $ind(Fr\{a, c\}) = 2$, $ind(Fr\{a, b, c\}) = ind(Fr\{a, c, e\}) = 0$. Then $indX \leq 3$.

Definition 5. (see [1]) A topological space is called zero-inductive dimensional if and only if there is a base for the open sets consisting of closed open (clopen) sets.

We introduce the following example.

Example 4. Let X be the set of real numbers, $\beta = \{(i, i + 1], i \in Z\}$ and $\tau = \{G : G = \underset{i}{\cup} B_i : B_i \in \beta\}$. Then X has a zero dimension. □

Proof. Since $(i, i+1]^c = (-\infty, i] \cup (i+1, \infty)$ is open. Thus every basic open set is both open and closed. Then every open set is closed and (X, τ) has a zero dimensional.

The following are examples for zero dimensional spaces.

Example 5. (see [1]) Let X be any non empty set with the discrete (indiscrete) topology, then the inductive dimension of X is zero.

Example 6. (see [1]) Let Q be the set of rational numbers with the topology inherited from the usual topology for the space of real numbers, then $indQ = 0$.

In the following, we have some results about inductive dimension.

Proposition 1. (see [1]) *Let X be a topological space. Then the followings are equivalent:*

(i) $indX \leq n$.

(ii) For every $x \in X$ *and every neighborhood* U *of* x, *there is an open set* V *such that* $x \in V \subset U$, *and* $indFrV \leq n-1$.

Theorem 1. (see [1]) *If X and Y are homeomorphic topological spaces, and if* $indX \leq n$, *then* $indY \leq n$.

Corollary 1. (see [1]) *If X and Y are homeomorphic topological spaces, then* $indX < indY$.

In the next section, the concept of semi inductive dimension will be introduced.

3. Semi Inductive Dimension

There are many scientific and engineering applications where detection of shape dimension from sample data is necessary. Topological dimensions of shapes are an important global feature of them.

For about half a century from 1914 to 1963, the class of open sets was used as the tool for most topological investigations. In 1963 Levine [11] introduced the concept of semi open sets. It is of interest to notice that several notions based on semi open sets are studied by many topologists, such as; Crossely [2, 3]. But, to the best of our knowledge this class of subsets is not applied in the context of dimension, the purpose of this article is to use semi open sets and its related notion to construct a new type of inductive dimensions.

Throughout this paper (X, τ) is a topological space, for $A \subset X, \overline{A}(A^0)$ is the closure (interior) of A with respect to τ.

Definition 6. (see [12]) A subset A of X is called a semi open set if there exists an open set V Such that $V \subset A \subset \bar{V}$, equivalently, A is semi open sets iff $A \subset A^{0-}$.

The class of all semi open sets of X is denoted by SO(X).

3.1. Semi Open Sets in Digital Topology

The term digital topology entered the human knowledge in the sixties of twentieth century to study connectedness of images on computer screen. The first approach for digital topology was depending on graph theory, by defining types of connectedness and paths. Some paradoxes, known by connectivity paradoxes resulted [11]. The solution for these paradoxes by the emergence of topological structure instead of graph theory concepts and this is known by *topological approach to digital topology.*

In the following, the digital topology on integers is explained [11]

Let $Z = \{0, \pm 1, \pm 2, \pm 3, ...\}$ be the set of integers and S is subbase, where $S = \{\{2n-1, 2n, 2n+1\} : n \in Z\}$, then the base β is

$$\beta = \{\{2n-1, 2n, 2n+1\} : n \in Z\} \cup \{2n+1 : n \in Z\}.$$

Then the digital topology τ is constructed as follows:

$$\tau = \{G : \text{ if } 2n \in G \text{ then } 2n+1, 2n-1 \in G\}.$$

The class of closed sets is:

$$\tau^c = \{F : \text{ if } \ 2n+1 \in F \text{ then } 2n, 2n+2 \in F\}.$$

Proposition 2. *A subset B of integers is semi open with respect to the digital topology if 2n ε B implies either 2n+1 ε B or 2n-1 ε B.*

Proof. If 2n ε B implies 2n+1 ε B, then B ^{0} contains thus B ^{0} is an open set contained in B and B is contained in its closure. Therefore B is semi open.

Conversely, if B is semi open and 2n ε B, then there exists an open set G and G $\subset$ B, B $\subset$ G $^-$ and hence B is a subset of every closed set containing G. Thus 2n does not belong to any open set contained in Z-G, consequently 2n ε G which is open.

Therefore 2n+1 ε B, 2n-1 ε B. □

Example 7. The set B={6,7}, $B^0 = \{7\}$, $B^{0-} = \{6, 7, 8\}$ and B is semi open. Also the set A={5,6} is semi open.

Definition 7. (see [12]) A subset $N_x \subset X$ is called a semi neighborhood of $x \in X$ if there exists a semi open set $U \subset X$ such that $x \in U \subset N_x$.

Remark 3. (see [12]) The intersection of two semi-open sets need not be semi-open sets.

Example 8. (see [12]) Let R be the set of real numbers with the Euclidean topology, then the two sets A = [a, b) and B = (c, a] are semi-open sets but A ∩ B={ a } is not semi open.

Definition 8. (see [12]) A point $p \in X$ is called a semi boundary point of $W \subset X$ if every semi- open set containing p meet W and X-W. The set W $_b$ of all semi- boundary points of W is called "the semi boundary set of W".

Definition 9. (see [3]) Let f : X → Y be a mapping, then f is called semi-continuous if the inverse image of each open set in Y is semi open in X.

Definition 10. (see [2]) A mapping f : X → Y is called M - semi continuous, if the inverse image of each semi open set in Y is semi- open set in X.

Definition 11. (see [3]) A mapping f : X → Y is called pre semi open, if the image of each semi -open is semi open.

Definition 12. A spaces X and Y are said to be semi- homeomorphic, if there exists a mapping f : X → Y such that f is

(i) Bijective,

(ii) M - semi continuous,

(iii) Pre semi open.

Such f is called a semi homeomorphisms.

In the following, we give the definition of semi- inductive dimension.

Definition 13. Let (X, τ) be a topological space, then (X, τ) has semi inductive dimensional -1 iff X= ∅.

Definition 14. Let (X, τ) be a topological space and let n be an integer greater than or equal zero. Then (X, τ) has semi inductive dimension ≤ n if it has a base β such that for every $B \in \beta$, the semi boundary of B (S.Fr B) has semi-inductive dimension ≤ n-1.

If X has semi -inductive dimension ≤ n and if it is false that X has semi-inductive dimension ≤ n-1. Then the semi- inductive dimension of X is n, and we write (S.ind X = n).

If for every $n \in Z_+$ it is false that X has semi- inductive dimension ≤ n. Then X is said to be infinite semi – inductive dimension, and we write S. ind X = ∞.

Example 9. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Then S.ind X = 0, ind X = 0.

Example 10. Let $X = \{a, b, c, d\}$,

$$\beta = \{\{a\}, \{a, b\}, \{b, c\}, \{b\}\}$$

and $\tau = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}, \{b\}, \{a, b, c\}\}$

Then S. ind X $\leq$ 2, ind X $\leq$ 2.

Example 11. Let $X = \{a, b, c, d, e\}$, $\beta = \{\{a, b, c\}, \{b, d\}, \{b, c, e\}, \{b\}, \{b, c\}\}$,

$$\tau = \left\{ \begin{array}{l} \phi, X, \{a, b, c\}, \{b, d\}, \{b, c, e\}, \{b\}, \{b, c\}, \{a, b, c, d\}, \{a, b, c, e\}, \\ \{b, c, d, e\}, \{b, d, c\} \end{array} \right\}.$$

Then S.ind X $\leq$ 3.

The following example indicates that semi inductive dimensions and inductive dimensions are not generally equal.

Example 12. Let $X = Z = \{......, -2, -1, 0, 1, 2, ...\}$ and $\beta = \{\{2n+1\}, \{2n-1, 2n, 2n+1\} : n \in Z\}$. We have $A \in SO(X, \tau)$ iff 2 n $\in$ A implies 2 n - 1 $\in$ A or 2 n + 1 $\in$ A. Then ind X =1, but S. ind X = 0.

Proposition 3. *Let (X, τ) be a topological space, then S.ind X $\leq$ ind X.*

Proof. Since S.F $_r$ (B) $\leq$ F $_r$ (B) for all B $\subset$ X, then the proof is obvious. □

Proposition 4. *Let X be a topological space, then the followings are equivalent:*

(i) S. ind X $\leq$ n,

(ii) For every x $\in$ X and every open neighborhood U of x,

There is a semi open set $V(x, U)$ such that x $\in V(x, U) \subset$ U and S.ind (S.Fr $V(x, U)) \leq$ n-1.

Proof. If U is an open neighborhood of $x \in X$, then there exist a basic open set $V(x, U)$ (belong to the base β of the topology on X) such that x $\in V(x, U) \subset$ U. Since S. ind X $\leq$ n then S.ind (S.Fr $V(x, U)$) $\leq$ n-1. Conversely, if (ii) is true, the class $\{V(x, U) : x \in X\}$ is a base for the topology on X. Since S.ind (S.Fr $V(x, U)$) $\leq$ n-1, then S. ind X $\leq$ n. □

Proposition 5. (see [2]) *Let X be a topological space, and suppose A $\subset$ X. If B $\subset$ X then S. Fr $_A$ (B $\cap$ A) $\subset$ A $\cap$ S. Fr $_X$ B.*

Proof. If p $\in$ S. Fr $_A$ (B $\cap$ A) then p $\in$ A. Now let U be an X semi neighborhood of p, then U $\cap$ A is an A semi neighborhood of p , and so it meets both B $\cap$ A and A - B because p $\in$ S. Fr $_A$)B $\cap$ A). This implies that U meets both B and X - B proving that p $\in$ S. Fr $_X$ B. Thus S. Fr $_A$) B $\cap$ A) $\subset$ A $\cap$ S. Fr $_X$ B. □

Theorem 2. *A Semi homeomorphism f: X $\rightarrow$ Y has the following properties :*

(i) S. CL (f^{-1} (B)) = f^{1-}) S.CL B) for every $B \subset Y$.
(ii) S. CL (f (B)) = f (S. CL B) for every $B \subset X$.
(iii) S. int (f^{1-} B)) = f^{-1}) S. int B) for every $B \subset Y$.
(iv) S. int (f (B)) = f (S. int B) for every $B \subset X$.

Proposition 6. *Let $f : X \to Y$ is a semi -homomorphism mapping, then S.Fr (f (B)) = f (S. Fr (B)) for every B is element of base of X.*

Proof. Since S. Fr (A) = S. CL A)) - S. int (A)) for every A, from the terms (ii), (iv) in the theorem, then the proof is complete. □

Theorem 3. *If X and Y are semi homomorphic topological spaces and if S. ind $X \leq n$ then S. ind $Y \leq n$.*

Proof. Let f : X → f(X) = Y be homomorphism. If β is a base for the topology of X, then f(β)= ({f (B) : B $\in \beta$ } is a base for the topology of Y, and for each B we have S. Fr f(B)=f(S. Fr (B)). We now use induction, the theorem is true for n = 0, for if β is a base for the topology of X such that S. Fr B = for each B $\in \beta$ then f (B (is a base of the topology of Y, and S. Fr (f(B) = f (S. Fr B) = for each B.

Assume the theorem is true for n-1, and suppose S. ind X $\leq$ n then there is a base β for X such that S. ind Fr B $\leq$ n-1 for each B $\in \beta$.

Again f (β) = {f (B) : B $\in \beta$ } is a base for Y, and S. Fr (f (B)) = f (S. Fr B), thus for any B $\in \beta$, the sets S. Fr (f (B)) and S. Fr (B) are homomorphic.

Thus, S. ind Fr (f (B)) $\leq$ n-1, this proves that S. ind Y $\leq$ n. □

Theorem 4. *Suppose X is a topological space with S. ind $X \leq n$. If $A \subset X$ is open, then S. ind $A \leq n$.*

Proof. We will use induction, the theorem is obviously true for n = -1. Assume it to be true for n-1, and suppose S. ind X $\leq$ n, then there is a base β for the topology of X with S. ind S. Fr $_X$ B $\leq$ n-1 for each B $\in \beta$. Now β_A = {B $\cap$ A : B $\in \beta$ } is a base for the subspace topology of A. From Proposition 1.4.2, we know that S. Fr $_A$ (B $\cap$ A) $\subset$ A $\cap$ S. Fr $_X$ B $\subset$ S. Fr $_X$ B for each B $\in \beta$. Thus S. ind Fr $_A$ (B $\cap$ A) $\leq$ n-1, from the induction hypothesis, which proves that S. ind A $\leq$ n. □

4. Conclusion

The algorithm presented in this work for computing topological dimension is simple and easy for application fields, it is applicable for any data set classified according overlapping granules. So it can be used in determining reducts and cores of information systems. Semi inductive dimensions help to construct types of zero dimensional spaces from nonzero dimensional ones which helps us in increasing the accuracy of computations.

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