International Journal of Pure and Applied Mathematics

Volume 41 No. 5 2007, 667-677

THE CYCLE-COMPLETE GRAPH RAMSEY NUMBER $r(C_8, K_7)$

M.M.M. Jaradat[1 §], A.M.M. Baniabedalruhman[2]

[1,2]Department of Mathematics
Yarmouk University
Irbid, JORDAN
[1]e-mails: mmjst4@yu.edu.jo
[2]e-mails: ahmad12982@yahoo.com
[1]Department of Mathematics and Physics
Qatar University
Doha, QATAR

Abstract: The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that every graph G of order N contains a cycle C_m on m vertices or has independent number $\alpha(G) \geq n$. It has been conjectured by Erdös, Faudree, Rousseau and Schelp that $r(C_m, K_n) = (m-1)(n-1)+1$ for all $m \geq n \geq 3$ (except $r(C_3, K_3) = 6$). This conjecture holds for $n = 3, 4, 5, 6$. In this paper we will present a proof for the conjecture in the case $n = 8$ and $m = 7$.

AMS Subject Classification: 05C55, 05C35
Key Words: Ramsey number, independent set, cycle graph, complete graph

1. Introduction

Through out this paper we adopt the standard notations C_m and K_n for the cycle and complete on m and n vertices, respectively. The minimum degree and the complement of a graph G is denoted by $\delta(G)$ and $\bar{G}$, respectively.

Received: July 29, 2007

[§]Correspondence address: Department of Mathematics and Physics, Qatar University, Doha, QATAR

The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that for every graph G of order N contains C_m or $\bar{G}$ contains K_n. In one of the earliest contributions to graphical Ramsey theory, Bondy and Erdös [3] proved the following result.

Theorem 1.1. (Bondy and Erdös) *For all* $m \geq n^2 - 2, r(C_m, K_n) = (m-1)(n-1)+1$.

The restriction in Theorem 1.1 was improved by Schiermeyer [11] when he showed that the equality holds for $n \geq 5$ with $m \geq n^2 - 2n$ and by Nikiforov [8] when he proved the equality for $m \geq 4n+2$. Erdös et al [4] gave the following conjecture.

Conjecture. $r(C_m, K_n) = (m-1)(n-1)+1$, *for all* $m \geq n \geq 3$ *except* $r(C_3, K_3) = 6$.

The conjecture was confirmed by Faudree and Schepl [5] and Rosta [10] for $n = 3$ in early work on Ramsey theory. Sheng al [13] and Bollobás et al [2] proved the conjecture for $n = 4$ and $n = 5$, respectively. Recently, the conjecture was proved by Schiermeyer [11] for $n = 6$. Most recently, Baniabedalruhman [1] proved that $r(C_7, K_7) = 37$. In a related work, Radziszowski and Tse [9] showed that $r(C_4, K_7) = 22$ and $r(C_4, K_8) = 26$. In [7] Jayawardene and Rousseau proved that $r(C_5, K_6) = 21$. Also, Schiermeyer [12] proved that $r(C_5, K_7) = 25$. In this article we will prove the conjecture for the case $n = 8$ and $m = 7$.

For completeness, we recall the following definitions: an independent set of vertices of a graph G is a subset of $V(G)$ in which no two vertices are adjacent. The independence number of a graph $G, \alpha(G)$, is the size of the largest independent set. The neighbor of the vertex u is the set of all vertices of G that are adjacent to u, denoted by $N(u)$. $N[u]$ denote to $N(u) \cup \{u\}$. Suppose that $V_1 \subseteq V(G)$ and V_1 is a non empty, the subgraph of G whose vertex set is V_1 and whose edge set is the set of those edges of G that have both ends in V_1 is called the subgraph of G induced by V_1, denoted by $\langle V_1 \rangle_G$.

2. The Main Results

The following lemma of Sheng et al [13] will be used frequently in proving the result.

Lemma 2.1. *Suppose* G *contains the cycle* $u_1u_2...u_{n-1}u_1$ *of length* $n-1$ *but no cycle of length* n. *Let* $X \subseteq V(G) \backslash \{u_1, u_2, ..., u_{n-1}\}$. *Then:*

(a) No vertex $x \in X$ is adjacent to two consecutive vertices on the cycle.

(b) If $x \in X$ is adjacent to u_i and u_j, then $u_{i+1}u_{j+1} \notin E(G)$.

(c) If $x \in X$ is adjacent to u_i and u_j, then no vertex $y \in X$ is adjacent to both u_{i+1} and u_{j+2}.

Lemma 2.2. *Let G be a graph of order greater than or equal 43 that contains neither C_8 nor a 7-elements independent set. Then $\delta(G) \geq 7$.*

Proof. Suppose that G contains a vertex of degree less than 7, say u. Since $r(C_8, K_6) = 36$. Then $|V(G) - N[u]| \geq 36$, as a result $G - N[u]$ has an independent set consisting of 6 vertices. This set with the vertex u is an independent set consisting of 7 vertices. This is a contradiction. The proof the lemma is complete. □

Lemma 2.3. *Let G be a graph with minimum degree $\delta(G) \geq 7$, that contains neither C_8 nor a 7-element independent set. If G contains K_7, then $|V(G)| \geq 56$.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the vertex set of K_7. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 7$. Since $\delta(G) \geq 7$, $U_i \neq \emptyset$ for all $1 \leq i \leq 7$. Since there is a path of order 7 joining any two vertices of U, as a result $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 7$ (otherwise, if $w \in U_i \cap U_j$ for some $1 \leq i < j \leq 7$, then the concatenation of the u_i-u_j-path of order 7 with $u_i w u_j$, is a cycle of order 8, a contradiction). Similarly, since there is a path of order 6 joining any two vertices of U, as a result for all $1 \leq i < j \leq 7$ and for all $x \in U_i$ and $y \in U_j$, $xy \notin E(G)$ (otherwise, if there is $1 \leq i < j \leq 7$ such that $x \in U_i$, $y \in U_j$ and $xy \in E(G)$, then the concatenation of the u_i-u_j-path of order 6 with $u_i x y u_j$ is a cycle of order 8, a contradiction). Also, since there is a path of order 5 joining any two vertices of U, as a result, $N_R(U_i) \cap N_R(U_j) = \emptyset$, $1 \leq i < j \leq 7$ (otherwise, if there are $1 \leq i < j \leq 7$ such that $w \in N_R(U_i) \cap N_R(U_j)$, then the concatenation of the $u_i u_j$- path of order 5 with $u_i x w y u_j$, is a cycle of order 8 where $x \in U_i$, $y \in U_j$ and $xw, wy \in E(G)$, a contradiction). Therefore, $|U_i \cup N_R(U_i) \cup \{u_i\}| \geq \delta(G) + 1$. Thus, $|V(G)| \geq 7(\delta(G) + 1) \geq (7)(8) = 56$. The proof of the lemma is complete. □

Lemma 2.4. *Let G be a graph with minimum degree $\delta(G) \geq 7$, that contains neither C_8 nor a 7-elements independent set. If G contains $K_7 - S_5$, then $|V(G)| \geq 48$.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the vertex set of $K_7 - S_5$ where the induced subgraph of $\{u_1, u_2, u_3, u_4, u_5, u_6\}$ is isomorphic to K_6. Without loss of generality we may assume that u_1u_7, $u_2u_7 \in E(G)$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 7$. Then, as in Lemma 2.3, $U_i \neq \emptyset$

for all $1 \leq i \leq 7$. Note that, $U_i \cap U_j = \emptyset$ for all $1 \leq i < j \leq 7$ except possibly for $i = 1$ and $j = 2$ (suppose that there are $1 \leq i < j \leq 7$ such that $w \in U_i \cap U_j$ and $(i, j) \neq (1,2)$. Then we consider the following: 1) $i = 1$ and $j = 5$. Then $u_5wu_1u_7u_2u_3u_4u_6u_5$ is a C_8, a contradiction. 2) $i = 1$ and $j = 7$. Then $u_7wu_1u_6u_5u_4u_3u_2u_7$ is a C_8, a contradiction. 3) $i = 3$ and $j = 7$. Then $u_7wu_3u_2u_5u_4u_6u_1u_7$ is a C_8, a contradiction. 4) $i = 4$ and $j = 5$. Then $u_4wu_5u_6u_1u_7u_2u_3u_4$ is a C_8, a contradiction. 5) i,j are not as in the above cases. Then by similarly arguments as in (1), (2), (3) and (4), G contains a C_8. This is a contradiction). Also, for all $1 \leq i < j \leq 7$ and for any $x \in U_i$ and $y \in U_j$, we have $xy \notin E(G)$ (suppose that there are $1 \leq i < j \leq 7$, $x \in U_i$ and $y \in U_j$ such that $xy \in E(G)$. Then we consider the following: 1) $i = 1$ and $j = 5$. Then $xu_1u_2u_3u_4u_6u_5yx$ is a C_8, a contradiction. 2) $i = 1$ and $j = 7$. Then $xu_1u_3u_4u_5u_2u_7yx$ is a C_8, a contradiction. 3) $i = 3$ and $j = 7$. Then $xu_3u_4u_5u_1u_2u_7yx$ is a C_8, a contradiction. 4) $i = 4$ and $j = 5$. Then $xu_4u_3u_2u_1u_6u_5yx$ is a C_8, a contradiction. 5) i,j are not as in the above cases. Then by similarly arguments as in (1), (2), (3) and (4), G contains a C_8. This is a contradiction). Moreover, $N_R(U_i) \cap N_R(U_j) = \emptyset$, $1 \leq i < j \leq 7$ except possibly for $i = 1$ and $j = 2$ (suppose that there are $1 \leq i < j \leq 7$ such that $w \in N_R(U_i) \cap N_R(U_j)$ and $(i,j)\neq(1,2)$. Then we consider the following: 1) $i = 1$ and $j = 5$. Let $x \in U_1$, $y \in U_5$ and $xw, yw \in E(G)$. Then $xwyu_5u_4u_3u_2u_1x$ is a C_8, a contradiction. 2) $i = 1$ and $j = 7$. Let $x \in U_1$, $y \in U_7$ and $xw, yw \in E(G)$. Then $xwyu_7u_2u_3u_4u_1x$ is a C_8, a contradiction. 3) $i = 3$ and $j = 7$. Let $x \in U_3$, $y \in U_7$ and $xw, yw \in E(G)$. Then $xwyu_7u_1u_2u_4u_3x$ is a C_8, a contradiction. 4) i,j are not as in the above cases. Then by similarly arguments as in (1), (2) and (3), G contains a C_8. This is a contradiction). Therefore $|U_i \cup N_R(U_i) \cup \{u_i\}| \geq \delta(G) + 1$ for all $2 \leq i \leq 7$. Thus, $|V(G)| \geq 6(\delta(G) + 1) \geq (6)(8) = 48$. The proof of the lemma is complete. □

Lemma 2.5. *Let G be a graph with minimum degree, $\delta(G) \geq 7$, that contains neither C_8 nor a 7-element independent set. If G contains K_6, then G contains $K_7 - S_5$ or K_7.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the vertex set of K_6. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 6$. Since $\delta(G) \geq 7$, $U_i \neq \emptyset$ for all $1 \leq i \leq 6$. We now consider the following cases.

Case 1. $U_i \cap U_j \neq \emptyset$ for some $1 \leq i < j \leq 6$, say $w \in U_i \cap U_j$. Then it is clear that G contains $K_7 - S_5$. In fact, the induced subgraph $< U \cup \{w\} >_G$ contains $K_7 - S_5$.

Case 2. $U_i \cap U_j = \emptyset$ for each $1 \leq i < j \leq 6$. Note that between any two vertices of U there are paths of order $4, 5$ and 6. Thus, as in Lemma 2.3 for all $1 \leq i < j \leq 6$, we have the following: 1) for all $x \in U_i$ and $y \in U_j$, $xy \notin E(G)$. 2) $N_R(U_i) \cap N_R(U_j) = \emptyset$. 3) for all $x \in N_R(U_i)$ and $y \in N_R(U_j)$, $xy \notin E(G)$.

Since $\alpha(G) \leq 6$, we have that the induced subgraphs $< U_i \cup N_R(U_i) >_G$, $1 \leq i \leq 6$ are complete graphs. Since $\delta(G) \geq 7$, as a result these complete graphs contain K_7. Hence, G contains K_7. The proof of the lemma is complete. □

Lemma 2.6. *Let G be a graph with minimum degree $\delta(G) \geq 7$, that contains neither C_8 nor a $7-$element independent set. If G contains $K_6 - S_4$, then G contains K_6.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the vertex set of $K_6 - S_4$ where the induced subgraph of $\{u_1, u_2, u_3, u_4, u_5\}$ is isomorphic to K_5. With out loss of generality we may assume that u_1u_6,$u_2u_6 \in E(G)$. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 6$. Then $U_i \neq \emptyset$ for all $1 \leq i \leq 6$ because $\delta(G) \geq 7$. Now we have the following two cases:

Case 1. $U_i \cap U_j \neq \emptyset$ for some $1 \leq i < j \leq 6$. We now consider the following subcases:

Subcase 1.1. $U_1 \cap U_6 \neq \emptyset$, say $u_7 \in U_1 \cap U_6$. Let $U_i^{/} = U_i - \{u_7\}$ for all $1 \leq i \leq 6$ and $R^{/} = R - \{u_7\}$. Then $U_i^{/} \neq \emptyset$ for all $1 \leq i \leq 6$ because $\delta(G) \geq 7$. Now, as in Lemma 2.5, $U_i^{/} \cap U_j^{/} = \emptyset$ for all $2 \leq i < j \leq 6$ because otherwise G contains C_8, a contradiction. Also, for all $2 \leq i < j \leq 6$ and for any $x \in U_i^{/}$ and $y \in U_j^{/}$ we have that $xy \notin E(G)$ because otherwise G contains C_8, a contradiction. Moreover, for all $2 \leq i < j \leq 6$, $N_{R^{/}}(U_i^{/}) \cap N_{R^{/}}(U_j^{/}) = \emptyset$ because otherwise G contains C_8, a contradiction. Also, For all $2 \leq i < j \leq 6$, and for all $x \in N_{R^{/}}(U_i^{/})$ and $y \in N_{R^{/}}(U_j^{/})$ we have that $xy \notin E(G)$ because otherwise G contains C_8, a contradiction..

Now, since $\alpha(G) \leq 6$, as a result there are at least two induced subgraph $< U_i^{/} \cup N_{R^{/}}(U_i^{/}) >_G$, $2 \leq i \leq 6$ are complete. Since $\delta(G) \geq 7$, it implies that these complete graphs contain K_6. Hence, G contains K_6.

Subcase 1.2. $U_3 \cap U_6 \neq \emptyset$, say $u_7 \in U_3 \cap U_6$. Let $U_i^{/} = U_i - \{u_7\}$ for all $1 \leq i \leq 6$ and $R^{/} = R - \{u_7\}$. Then $U_i^{/} \neq \emptyset$ for all $1 \leq i \leq 6$ because $\delta(G) \geq 7$. We now have the following I and II because otherwise G contains C_8: I) $U_i^{/} \cap U_j^{/} = \emptyset$ for all $1 \leq i < j \leq 5$. II) For all $1 \leq i < j \leq 5$ and for any $x \in U_i^{/}$ and $y \in U_j^{/}$ we have $xy \notin E(G)$. Note that between any two vertices of U there are paths of order 4 and 5. Thus, for all $1 \leq i < j \leq 5$, we have the following; 1) $N_{R^{/}}(U_i^{/}) \cap N_{R^{/}}(U_j^{/}) = \emptyset$. 2) For all $x \in N_{R^{/}}(U_i^{/})$ and $y \in N_{R^{/}}(U_j^{/})$ we have $xy \notin E(G)$.

Since $\alpha(G) \leq 6$, as a result there are at least four induced subgraph $< U_i^{/} \cup N_{R^{/}}(U_i^{/}) >_G$, $1 \leq i \leq 5$ are complete. Since $\delta(G) \geq 7$, it implies that these complete graphs contain K_6. Hence, G contains K_6.

Subcase 1.3. $U_2 \cap U_3 \neq \emptyset$, say $u_7 \in U_2 \cap U_3$. Let $U^{/} = U \cup \{u_7\}$ and $R^{/} = G - U^{/}$ and $U_i^{/} = U^{/} \cap R^{/}$. Then $U_i^{/} \neq \emptyset$ for all because $\delta(G) \geq 7$. We now have the following I and II because otherwise G contains C_8: I) $U_i^{/} \cap U_j^{/} = \emptyset$ for all $i < j$ where $i, j = 1, 4, 5, 6, 7$. II) For all $i, j \in \{1, 4, 5, 6, 7\}$ such that $i < j$ and for any $x \in U_i^{/}$ and $y \in U_j^{/}$ we have $xy \notin E(G)$.

Note that between any two vertices of $\{u_1, u_4, u_5, u_6, u_7\}$ there are paths of order 4 and 5. Thus, for all $i, j \in \{1, 4, 5, 6, 7\}$ such that $i < j$, we have the following: 1) $N_{R^{/}}(U_i^{/}) \cap N_{R^{/}}(U_j^{/}) = \emptyset$. 2) For all $x \in N_{R^{/}}(U_i^{/})$ and $y \in N_{R^{/}}(U_j^{/})$ we have $xy \notin E(G)$.

Since $\alpha(G) \leq 6$, as a result there are at least one induced subgraph $< U_i^{/} \cup N_{R^{/}}(U_i^{/}) >_G, i = 6, 7$ is complete graphs. Since $\delta(G) \geq 7$ and for all $x \in N(U_i^{/}) - \{u_i\}, i = 6, 7$, we have that $x \notin U^{/}$. It implies that these complete graphs contain K_6. Hence, G contains K_6.

Subcase 1.4. $U_i \cap U_j \neq \emptyset$ and i,j are not as in the above subcases. Then by using the same arguments as in Subcase 1, Subcase 2 and Subcase 3, G contains K_6.

Case 2. $U_i \cap U_j = \emptyset$ for all $2 \leq i < j \leq 6$. Note that between any two vertices of $U - \{u_1\}$ there are paths of order $4, 5$ and 6. ($u_2u_3u_4u_5u_1u_6$ is a path of order 6 between u_2 and $u_6u_3u_4u_5u_1u_2u_6$ is a path of order 6 between u_3 and u_6. Similarly, we can construct paths for the other cases). Thus, for all $2 \leq i < j \leq 6$ we have the following: 1) For all $x \in U_i$ and $y \in U_j$ we have $xy \notin E(G)$. 2) $N_R(U_i) \cap N_R(U_j) = \emptyset$. 3) For all $x \in N_R(U_i)$ and $y \in N_R(U_j)$ we have $xy \notin E(G)$.

Since $\alpha(G) \leq 6$, as a result there are at least four induced subgraph $< U_i \cup N_R(U_i) >_G$, $2 \leq i \leq 6$ are complete. Since $\delta(G) \geq 7$, it implies that these complete graphs contain K_6. Hence, G contains K_6. The proof the Lemma is complete. □

Lemma 2.7. *Let G be a graph with minimum degree $\delta(G) \geq 7$, that contains neither C_8 nor a $7-$element independent set. If G contains K_5, then G contains $K_6 - S_4$ or K_6.*

Proof. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be the vertex set of K_5. Let $R = G - U$ and $U_i = N(u_i) \cap V(R)$ for each $1 \leq i \leq 5$. Since $\delta(G) \geq 7$, $|U_i| \geq 3$ for all $1 \leq i \leq 5$. We now split our work into the following two cases.

Case 1. There are $1 \leq i < j \leq 5$ such that $U_i \cap U_j \neq \emptyset$. Then G contains $K_6 - S_4$. The result is obtained.

Case 2. $U_i \cap U_j = \emptyset$ for each $1 \leq i < j \leq 5$. We split this case into two subcases.

Subcase 2.a. There is u_i-u_j-path of order 4 in $G - U$, say $i = 1$, $j = 2$ and the path is $u_1u_6u_7u_2$. For simplicity, in the rest of this subcase we consider $U_i^{/} = N(U_i) \cap V(R^{/})$ where $R^{/} = G - (U \cup \{u_6,u_7\})$. Then $U_i^{/} \cap U_j^{/} = \emptyset$ for each $3 \leq i < j \leq 7$. Since between any two vertices of $\{u_3, u_4, u_5, u_6, u_7\}$ there is a path of order 6, as a result for any $x \in U_i^{/}$ and $y \in U_j^{/}$ we have $xy \notin E(G)$ for all $3 \leq i < j \leq 7$. Since $\alpha(G) \leq 6$, there are at least one induced subgraph $< U_i^{/} >_G$, $6 \leq i \leq 7$ is a complete graph. Note that u_6 is adjacent only to u_1 and u_7 of $U \cup \{u_6, u_7\}$ and u_7 is adjacent only to u_2 and u_6 of $U \cup \{u_6,u_7\}$. Since $d(u_6)$, $d(u_7) \geq 7$, we have $d_{R^{/}}(u_6)$, $d_{R^{/}}(u_7) \geq 5$. Thus, $< U_i^{/} >_G$ contains K_6 and hence G contains K_6.

Subcase 2.b. There is no u_i-u_j-path of order 4 in $G-U$, that is for any $x \in U_i$ and $y \in U_j$, $xy \notin E(G)$ for each $1 \leq i < j \leq 5$. Since between any two vertices of U there are paths of order 4 and 5, as a result $N_R(U_i) \cap N_R(U_j) = \emptyset$ and for any $x \in N_R(U_i)$ and $y \in N_R(U_j)$ we have $xy \notin E(G)$ for each $1 \leq i < j \leq 5$. Since $\alpha(G) \leq 6$, as a result there are at least four induced subgraph $< U_i \cup N_R(U_i) >_G$, $1 \leq i \leq 5$ are complete. Since $\delta(G) \geq 7$, it implies that these complete graphs contain K_6. Hence, G contains K_6. The proof of the lemma is complete. □

Theorem 2.8. $r(C_8, K_7) = 43$.

Proof. Consider $G = 6K_7$. Then $\alpha(G) = 6$ and it does not contain C_8. Thus, $r(C_8, K_7) \geq 43$. Now, suppose that there exist a graph G of order 43 that contains neither C_8 nor a 7-element independent set. Now we consider the following two cases:

Case 1. G contains K_5. Then by Lemmas 2.7, 2.6, 2.5, 2.4, 2.3, 2.2, $|G| \geq 48$. This is a contradiction.

Case 2. G does not contain K_5. Since $r(C_7, K_7) = 37$, G contains C_7. Let $C_7 = u_1u_2u_3u_4u_5u_6u_7u_1$ and $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$. Since $r(C_8, K_6) = 36$ and $|V(G) - U| = 36$, $G - U$ contains 6-element independent set. Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be a 6-element independent set. Since $\alpha(G) \leq 6$, each vertex on U is adjacent to at least one vertex in V. Since $|U| = 7$ and $|V| = 6$, at least one vertex of V is adjacent to two or more vertices of U. But by Lemma 2.1 each vertex of V is adjacent to at most three vertices of U. We now consider two subcases.

Subcase 2.a. There exist a vertex of V adjacent to three vertices of U, say

v_1 is adjacent to u_1, u_3, u_5. Let v_2 is adjacent to u_6. Then by Lemma 2.1 v_2 does not adjacent to u_2. Let v_3 is adjacent to u_2. Then $d_{G-U}(v_1), d_{G-U}(v_2), d_{G-U}(v_3) \geq 4$, because $\delta(G) \geq 7$. To this end, we have the following because otherwise G contains C_8: I) $N_{G-U}(v_i) \cap N_{G-U}(v_j) = \emptyset$ for all $1 \leq i < j \leq 3$. II) For each $x \in N_{G-U}(v_i)$ and $y \in N_{G-U}(v_j)$, we have $xy \notin E(G)$ for all $1 \leq i < j \leq 3$. III) For each $x \in N_{G-U}(v_i), 1 \leq i \leq 3$, we have that $xu_1 \notin E(G)$.

Since G does not contain K_5, then $N_{G-U}(v_i)$ is not complete for each $1 \leq i \leq 3$. Hence, $\alpha(\langle N_{G-U}(v_i)\rangle_G) \geq 2, 1 \leq i \leq 3$. This implies that $\alpha(\left\langle \cup_{i=1}^{3} N_{G-U}(v_i)\right\rangle_G) \geq 2+2+2 = 6$ and hence $\alpha(\left\langle \cup_{i=1}^{3} N_{G-U}(v_i) \cup \{u_1\}\right\rangle_G) \geq 6+1 = 7$. Thus $\alpha(G) \geq 7$. This is a contradiction.

Subcase 2.b. There is no vertex of V which is adjacent to three vertices of U. Then there is a vertex which adjacent to two vertices of U, say v_1. □

Subsubcase 2.b.1. v_1 adjacent to u_1 and u_4. We have the following subsubsubcases:

Subsubsubcase 2.b.1.a. v_2 adjacent to u_5 and v_3 adjacent to u_7. To this end we have the following because otherwise G contains C_8: I) $N_{G-U}(v_i) \cap N_{G-U}(v_j) = \emptyset$ for all $1 \leq i < j \leq 3$. II) For all $x \in N_{G-U}(v_1)$ and $y \in N_{G-U}(v_j), j = 2, 3$ we have that $xy \notin E(G)$. III) For all $x \in N_{G-U}(v_i)$, $1 \leq i \leq 3$ we have that $xu_6 \notin E(G)$. Now, we have the following claim:

Claim. $\alpha(\langle N_{G-U}(v_2) \cup N_{G-U}(v_3) \cup \{v_2, v_3\}\rangle_G) \geq 4$.

Proof of Claim. We consider the following:

1) $\alpha(\langle N_{G-U}(v_i)\rangle_G) \geq 3$ for some $i = 2, 3$, say $i = 2$. Then $\alpha(\langle N_{G-U}(v_2) \cup\{v_3\}\rangle_G) \geq 4$.

2) $\alpha(\langle N_{G-U}(v_i)\rangle_G) \leq 2$ for each $2 \leq i \leq 3$. Since G does not contain K_5, $\alpha(\langle N_{G-U}(v_i)\rangle_G) = 2$ for each $2 \leq i \leq 3$. Since G does not contain K_5 and $|N_{G-U}(v_i)| \geq 5$ for all $2 \leq i \leq 3$, as a result $\langle N_{G-U}(v_i)\rangle_G$ has no isolated vertex. Hence, for all $x \in N_{G-U}(v_2)$ and $y \in N_{G-U}(v_3)$ we have $xy \notin E(G)$ (otherwise if $xy \in E(G)$, then x is not isolated vertex. Let $xw \in E(G)$. Then $yxwv_2u_5u_6u_7v_3y$ is a C_8, a contradiction). This implies $\alpha(\langle N_{G-U}(v_2) \cup N_{G-U}(v_3)\rangle_G) \geq \alpha(\langle N_{G-U}(v_2)\rangle_G) + \alpha(\langle N_{G-U}(v_3)\rangle_G) \geq 2+2 = 4$. The proof of the claim is complete. □

Now,

$$\begin{aligned}\alpha(\left\langle \cup_{i=1}^{3} N_{G-U}(v_i) \cup \{v_2, v_3, u_6\}\right\rangle_G) &= \alpha(\langle N_{G-U}(v_1)\rangle_G) \\ &+ \alpha\left(\langle N_{G-U}(v_2) \cup N_{G-U}(v_3) \cup \{v_2, v_3\}\rangle_G\right) + \alpha(\{u_6\}) \geq 2+4+1 = 7.\end{aligned}$$

This is a contradiction.

Subsubsubcase 2.b.1.b. v_2 adjacent to u_5 and u_7. Let v_3 adjacent to u_6.

Then, by Lemma 2.1, v_3 can not be adjacent to u_2 and u_3. Let v_4 adjacent to u_2 and v_5 adjacent to u_3. Then we have the following because otherwise G contains C_8: I) $N_{G-U}(v_i) \cap N_{G-U}(v_j) = \emptyset$ for all $3 \leq i < j \leq 5$. II) For all $x \in N_{G-U}(v_3)$ and $y \in N_{G-U}(v_i), i = 4, 5$ we have that $xy \notin E(G)$.

Since $\delta(G) \geq 7$, then $|N_{G-U}(v_i)| \geq 5$ for all $3 \leq i \leq 5$. Since G does not contain K_5, $\alpha(\langle N_{G-U}(v_i)\rangle_G) \geq 2$ for all $3 \leq i \leq 5$. We now consider the following:

1) $\alpha(\langle N_{G-U}(v_i)\rangle_G) \geq 3$ for some $4 \leq i \leq 5$, say $\alpha(\langle N_{G-U}(v_4)\rangle_G) \geq 3$. Then, by above (I), $\alpha(\langle N_{G-U}(v_4) \cup \{v_5\}\rangle_G) \geq 3 + 1 = 4$. Note that, by Lemma 2.1, $v_5u_4 \notin E(G)$. Also, for all $x \in N_{G-U}(v_i)$, $3 \leq i \leq 4$, we have that $xu_4 \notin E(G)$ (because otherwise if there is $x \in N_{G-U}(v_i)$, $3 \leq i \leq 4$ with $xu_4 \notin E(G)$. Then we have the following: 1) $i = 3$. Then $v_3xu_4u_3u_2u_1u_7u_6v_3$ is a C_8, a contradiction. 2) $i = 4$. Then $v_4xu_4u_5u_6u_7u_1u_2v_4$ is a C_8, a contradiction).

Therefore

$$\alpha(\langle \cup_{i=3}^{4} N_{G-U}(v_i) \cup \{v_5, u_5\}\rangle_G) = \alpha(\langle N_{G-U}(v_3)\rangle_G) + \alpha(\langle N_{G-U}(v_4) \cup \{v_5\}\rangle_G) + \alpha(\{u_4\}) \geq 2 + 4 + 1 = 7.$$

This is a contradiction.

2) $\alpha(\langle N_{G-U}(v_i)\rangle_G) = 2$ for all $4 \leq i \leq 5$. Since G does not contain K_5 and $|N_{G-U}(v_i)| \geq 5$ for all $3 \leq i \leq 5$, as a result $N_{G-U}(v_i)$ has no isolated vertex for all $3 \leq i \leq 5$. Now we have the following: I)For all $x \in N_{G-U}(v_4)$ and $y \in N_{G-U}(v_5)$ we have that $xy \notin E(G)$ (Suppose that there are $x \in N_{G-U}(v_4)$, $y \in N_{G-U}(v_5)$ with $xy \in E(G)$. Then $v_4uxywv_5u_3u_2v_4$ is a C_8, a contradiction where $u \in N_{G-U}(v_4), w \in N_{G-U}(v_5)$ with $xu, yw \in E(G)$ (because x, y are not isolated vertex in $\langle N_{G-U}(v_4)\rangle_G, \langle N_{G-U}(v_5)\rangle_G$, respectively)). II) For all $x \in N_{G-U}(v_i)$ and $3 \leq i \leq 5$ we have $xu_1 \notin E(G)$ because otherwise G contains C_8.

Therefore

$$\alpha(\langle \cup_{i=3}^{5} N_{G-U}(v_i) \cup \{u_1\}\rangle_G) = \alpha(\langle N_{G-U}(v_3)\rangle_G) + \alpha(\langle N_{G-U}(v_4)\rangle_G) + \alpha(\langle N_{G-U}(v_5)\rangle_G) + \alpha(\{u_1\}) \geq 2 + 2 + 2 + 1 = 7.$$

This is a contradiction.

Subsubcase 2.b.2. v_1 adjacent to u_1 and u_3. Then we consider the following.

Subsubsubcase 2.b.2.a. v_2 adjacent to u_4 and u_7. Then we have a similar case as in Subsubsubcase 2.b.1.b.

Subsubsubcase 2.b.2.b. v_2 adjacent to u_4 and v_3 adjacent to u_7. Then we have the following because otherwise G contains C_8: I) $N_{G-U}(v_i) \cap N_{G-U}(v_j) =$

$\emptyset$ for all $1 \leq i < j \leq 3$. II) For all $x \in N_{G-U}(v_i)$ and $y \in N_{G-U}(v_j), 1 \leq i < j \leq 3$, we have $xy \notin E(G)$.

Since $\delta(G) \geq 7$, $|N_{G-U}(v_i)| \geq 5$ for all $1 \leq i \leq 3$. Since G does not contain K_5, $\alpha(\langle N_{G-U}(v_i)\rangle_G) \geq 2$ for all $1 \leq i \leq 3$. We now have the following:

1) $\alpha(\langle N_{G-U}(v_i)\rangle_G) \geq 3$ for some $1 \leq i \leq 3$. Then $\alpha(\left\langle \cup_{i=1}^{3} N_{G-U}(v_i)\right\rangle_G) \geq 2+2+3=7$. This is a contradiction.

2) $\alpha(\langle N_{G-U}(v_i)\rangle_G) = 2$ for all $1 \leq i \leq 3$. Since G does not contain K_5 and $|N_{G-U}(v_i)| \geq 5$ for all $1 \leq i \leq 3, \langle N_{G-U}(v_i)\rangle_G$ has no isolated vertex for all $1 \leq i \leq 3$. Note that for all $x \in N_{G-U}(v_i), 1 \leq i \leq 2$, we have $xu_1, xu_6 \notin E(G)$ (Suppose that there are $x \in N_{G-U}(v_i), 1 \leq i \leq 2$ with $xu_1 \in E(G)$ or $xu_6 \in E(G)$. Then 1) $i = 1$. Then $v_1xu_1u_7u_6u_5u_4u_3v_1$, $v_1xu_6u_5u_4u_3u_2u_1v_1$ are C_8, a contradiction. 2) $i = 2$. Then $v_2xu_6u_7u_1u_2u_3u_4v_2$, $v_2wxu_1u_7u_6u_5\ u_4u_2$ are C_8, a contradiction where $w \in N_{G-U}(v_2)$ and $wx \in E(G)$, (because x is not isolated vertex)). To this end, we Consider the following:

2.a) $u_1u_6 \notin E(G)$. Then $\alpha(\left\langle \cup_{i=1}^{2} N_{G-U}(v_i) \cup \{u_1, u_6, v_3\}\right\rangle_G) \geq 2+2+3=7$. This is a contradiction.

2.b) $u_1u_6 \in E(G)$. Then for all $x \in N_{G-U}(v_i), 1 \leq i \leq 3$, we have $xu_2 \notin E(G)$ (Suppose that there are $x \in N_{G-U}(v_i), 1 \leq i \leq 3$ with $xu_2 \in E(G)$. Then we have the following: (1) $i = 1$. Then $v_1xu_2u_1u_6u_5u_4u_3v_1$ is a C_8, a contradiction. (2) $i = 2$. Then $v_2xu_2u_1u_7u_6u_5u_4v_2$ is a C_8, a contradiction. (3) $i = 3$. Then $v_3xu_2u_3u_4u_5u_6u_7v_3$ is a C_8, a contradiction). Therefor, $\alpha(\left\langle \cup_{i=1}^{3} N_{G-U}(v_i) \cup \{u_2\}\right\rangle_G) \geq 2+2+2+1=7$. This is a contradiction. And so the proof of the theorem is complete. ☐

References

[1] A. Baniabedalruhman, *On Ramsey Number for Cycle-Complete Graphs*, M.Sc. Thesis, Yarmouk University (2006).

[2] B. Bollobás, C.J. Jayawardene, Z.K. Min, C.C. Rousseau, H.Y. Ru, J. Yang, On a conjecture involving cycle-complete graph Ramsey numbers, *Australas. J. Combin.*, **22** (2000), 63-72.

[3] J.A. Bondy, P. Erdös, Ramsey numbers for cycles in graphs, *Journal of Combinatorial Theory, Series B*, **14** (1973), 46-54.

[4] P. Erdös, R.J. Faudree, C.C. Rousseau, R.H. Schelp, On cycle-complete graph Ramsey numbers, *J. Graph Theory*, **2** (1978), 53-64.

[5] R.J. Faudree, R.H. Schelp, All Ramsey numbers for cycles in graphs, *Discrete Mathematics*, **8** (1974), 313-329.

[6] G.R. Hendry, Ramsey numbers for graphs with five vertices, *Journal of Graph Theory*, **13** (1989), 245-248.

[7] C.J. Jayawardene, C.C. Rousseau, The Ramsey number for a cycle of length five versus a complete graph of order six, *J. Graph Theory*, **35** (2000), 99-108.

[8] V. Nikiforov, The cycle-complete graph Ramsey numbers, *Combin. Probab. Comput.,* **14**, No. 3 (2005), 349-370.

[9] S.P. Radziszowski, K.-K. Tse, A computational approach for the Ramsey numbers $r(C_4, K_n)$, *J. Comb. Math. Comb. Comput.*, **42** (2002), 195-207.

[10] V. Rosta, On a Ramsey type problem of J.A. Bondy and P. Erdös, I and II, *Journal of Combinatorial Theory, Series B,* **15** (1973), 94-120.

[11] I. Schiermeyer, All cycle-complete graph Ramsey numbers $r(C_n, K_6)$, *J. Graph Theory*, **44** (2003), 251-260.

[12] I. Schiermeyer, The cycle-complete graph Ramsey number $r(C_5, K_7)$, *Discussiones Mathematicae Graph Theory*, **25** (2005) 129-139.

[13] Y.J. Sheng, H.Y. Ru, Z.K. Min, The value of the Ramsey number $r(C_n, K_4)$ is $3(n-1)+1$ $(n \geq 4)$, *Ausralas. J. Combin.*, **20** (1999), 205-206.