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THE STRONG LAW OF LARGE NUMBERS ON A RANDOM GRAPH

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Abstract: Let $G(n,p)$ be a random graph on n labeled vertices $\{1,2,...,n\}$, where possible edge $\{i,j\}$ is present randomly and independently with the probability p, $0<p<1$. In this paper, we prove that the number of isolated vertices, the number of vertices of degree d, the number of isolated trees and the number of isolated copies of a fixed connected graph in a random graph $G(n,p)$ obey the strong law of large numbers.

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Key Words: strong law of large numbers, random graph, isolated vertices, vertex of degree d, isolated trees, isolated copies of connected graph

1. Introduction and Main Results

Let $X_1, X_2, ...$ be a sequence of random variables with finite expectations in a probability space and $S_n = \frac{1}{n}(X_1 + X_2 + ... + X_n)$. We say that the sequence (S_n) obeys the strong law of large numbers(SLLN) if

$$\frac{1}{n}[S_n - E(S_n)] \xrightarrow{a.s.} 0\,,$$

where a.s. stands for convergence almost surely. When the random variables are identically distributed, with the expectation μ, the law becomes:

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$$\frac{1}{n}S_n \xrightarrow{a.s.} \mu. \tag{1}$$

The strong law of large numbers (1) was original proved by Borel in the case of X_i's being independent Bernoulli random variables while the general form of (1) was proved by A.N. Kolmogorov. In our study, we investigate the SLLN of some sequences of dependent identically distributed random variables such as a sequence of indicator random variables, defined on a random graph; a collection of points or vertices, with lines or edges connecting pairs of them at random.

The study of random graphs has long history. A systematic study of random graphs began with the fluential work of Paul Erdös and Alfréd Rénnyi in the 1950s and 1960s (see [4]-[6]). The theory of random graphs lines at intersection between graph theory and probability theory, and studies the properties of typical random graphs. Different graph models produce different probability distribution on graphs.

Let $G(n,p)$ be a random graph on n labeled vertices $\{1,2,...,n\}$, where possible edge $\{i,j\}$ is present randomly and independently with the probability p, $0<p<1$. For each vertex i, if there are exactly d edges which incident to the vertex i, then it is said to have degree d and if $d=0$, then we say that the vertex i is isolated.

A tree in $G(n,p)$ is, by definition, a connected graph containing no cycles and it is isolated if there is no edge in $G(n,p)$ with one vertex in the tree and the other outside of the tree.

A subgraph K in $G(n,p)$ with vertex set $V(K)$ and edge set $E(K)$ is said to be isolated, if there is no edge from a vertex in $V(K)$ to another outside of $V(K)$ and it is isomorphic to a fixed graph H with vertex set $V(H)$ and edge set $E(H)$, if there is a bijection $f : V(H) \to V(K)$ such that $uv \in E(H)$ iff $f(u)f(v) \in E(K)$. We say that K is an isolated copy of H in $G(n,p)$ if K is an isolated subgraph in $G(n,p)$ and K is isomorphic to H.

From now on, we let p be a constant such that $0<p<1$ and $q=1-p$. Our main results below show that the number of isolated vertices, the number of vertices of degree d, the number of isolated trees and the number of isolated copies of a fixed graph in $G(n,p)$ obey the SLLN.

Theorem 1.1. *For each $n \in \mathbb{N}$ and $0 \le d \le n-1$, let S_n be the number of vertices of degree d in $G(n,p)$. Then S_n obeys the SLLN.*

Corollary 1.2. *The number of isolated vertices in $G(n,p)$ obeys the SLLN.*

Theorem 1.3. *For each $n \in \mathbb{N}$ and $2 \le k \le n$, let $S_{n,k}$ be the number of isolated trees of order k in $G(n,p)$. Then $S_{n,k}$ obeys the SLLN.*

Theorem 1.4. *Let H be a fixed connected graph and $S_{n,H}$ be the number of isolated copies of H in random graph $G(n,p)$. Then $S_{n,H}$ obeys the SLLN.*

2. Proof of Main Results

In our study, we use the following proposition as our tool.

Proposition 2.1. *Let $X, X_1, X_2, ...$ be random variables on a probability space $(\Omega, \Im, P)$ and $S_n = \sum_{i=1}^{n} X_i$. Then S_n obeys the SLLN if $\sum_{n=1}^{\infty} \frac{VarS_n}{n^2} < \infty$.*

Proof. Assume that $\sum_{n=1}^{\infty} \frac{VarS_n}{n^2} < \infty$. Let $\epsilon > 0$ be arbitrary and $A_n(\epsilon) = \left\{\omega \in \Omega \ : \ \left|\frac{S_n - ES_n}{n}(\omega)\right| > \epsilon\right\}$.

By Chebyšhev's inequality, we get

$$\sum_{n=1}^{\infty} P\left(A_n(\epsilon)\right) =: \sum_{n=1}^{\infty} P\left(\left|\frac{S_n - ES_n}{n}\right| > \epsilon\right) \leq \frac{1}{\epsilon^2}\sum_{n=1}^{\infty}\frac{VarS_n}{n^2} < \infty.$$

Hence by the first Borel-Cantelli Lemma, $P\left(\cap_n \cup_{m=n}^{\infty} A_m(\epsilon)\right) = 0$ ([3], pp 320). That is $P\left(\cup_n \cap_{m=n}^{\infty} A_m^c(\epsilon)\right) = 1$, for any $\epsilon > 0$.

Since $\cup_n \cap_{m=n}^{\infty} \left\{\omega \in \Omega \ : \ \left|\frac{S_n - ES_n}{n}(\omega)\right| \leq \epsilon\right\}$ is increasing with ϵ, we obtain

$$\begin{aligned}
&P\left(\left\{\omega \in \Omega \ : \ \frac{S_n - ES_n}{n}(\omega) \to 0\right\}\right) \\
&\quad = P\left(\left\{\omega \in \Omega : \ \forall \epsilon > 0, \ \exists n \geq 1, \ \forall m \geq n, \ \left|\frac{S_n - ES_n}{n}(\omega)\right| \leq \epsilon\right\}\right) \\
&\quad = P\left(\cap_{\epsilon>0} \cup_{n\geq 1} \cap_{m=n}^{\infty} \left\{\omega \in \Omega \ : \ \left|\frac{S_n - ES_n}{n}(\omega)\right| \leq \epsilon\right\}\right) \\
&\quad = \lim_{\epsilon \searrow 0} P\left(\cup_n \cap_{m=n}^{\infty} \left\{\omega \in \Omega \ : \ \left|\frac{S_n - ES_n}{n}(\omega)\right| \leq \epsilon\right\}\right) \\
&\quad = \lim_{\epsilon \searrow 0} P\left(\cup_n \cap_{m=n}^{\infty} A_m^c(\epsilon)\right) = 1.
\end{aligned}$$

That implies $\frac{1}{n}\left[S_n - ES_n\right] \xrightarrow{a.s.} 0$ as $n \to \infty$. □

Proof of Theorem 1.1. For each $i \in \{1, 2, ..., n\}$ and $0 \leq d \leq n-1$, let X_i be the indicator random variable defined as follows;

$$X_i = \begin{cases} 1, & \text{if the vertex } i \text{ has degree } d; \\ 0, & \text{otherwise.} \end{cases}$$

Set $S_n = \sum_{i=1}^{n} X_i$. Then S_n is the number of vertices of degree d in $G(n,p)$. Barbour, Karoński and Rucinski [1] show that

$$ES_n = nEX_i = n\binom{n-1}{d} p^d q^{n-1-d}, \tag{2}$$

and

$$\begin{aligned} VarS_n =& \frac{n}{n-1}\binom{n-1}{d}^2 (d-(n-1)p)^2 p^{2d-1} q^{2(n-d)-3} \\ &+ ES_n - \frac{1}{n}(ES_n)^2 . \end{aligned} \tag{3}$$

We will show that $\sum_{n=1}^{\infty} \frac{VarS_n}{n^2} < \infty$ by breaking into the case $d=0$ and $1 \le d \le n-1$.

For $d=0$, from (2) and (3), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{VarS_n}{n^2} &= \sum_{n=1}^{\infty} \left[\frac{\frac{n}{n-1}(n-1)^2 pq^{2n-3} + ES_n - n^{-1}(ES_n)^2}{n^2} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{n^2pq^{2n-3} - npq^{2n-3} + nq^{n-1} - nq^{2n-2}}{n^2} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{nq^{n-1} + n^2pq^{2n-3} - n(p+q)q^{2n-3}}{n^2} \right] \\ &= \sum_{n=1}^{\infty} \left[\frac{nq^{n-1} + n^2pq^{2n-3} - nq^{2n-3}}{n^2} \right] = \frac{1}{q}\sum_{n=1}^{\infty} \frac{q^n}{n} + \frac{p}{q^3}\sum_{n=1}^{\infty} q^{2n} - \frac{1}{q^3}\sum_{n=1}^{\infty} \frac{q^{2n}}{n}. \end{aligned}$$

Each series on the right side converges by the ratio test, which implies $\sum_{n=1}^{\infty} \frac{VarS_n}{n^2} < \infty$.

Suppose that $1 \le d \le n-1$. It follows from (2)-(3) that

$$\begin{aligned} VarS_n &= \frac{n}{n-1}\binom{n-1}{d}^2 [d-(n-1)p]^2 p^{2d-1} q^{2(n-d)-3} + ES_n - \frac{1}{n}(ES_n)^2 \\ &= \frac{n}{n-1}\binom{n-1}{d}^2 [d-(n-1)p]^2 p^{2d-1} q^{2(n-d)-3} \\ &\quad + n\binom{n-1}{d} p^d q^{n-1-d} - n\binom{n-1}{d}^2 p^{2d} q^{2(n-d)-2} \\ &= \frac{n}{d}\binom{n-1}{d}\binom{n-2}{d-1}\left[d^2 - 2dp(n-1) + p^2(n-1)^2\right] p^{2d-1} q^{2(n-d)-3} \\ &\quad + n\binom{n-1}{d} p^d q^{n-1-d} - n\binom{n-1}{d}^2 p^{2d} q^{2(n-d)-2} \\ &= nd\binom{n-1}{d}\binom{n-2}{d-1} p^{2d-1} q^{2(n-d)-3} \end{aligned}$$

$$
\begin{aligned}
&\qquad - 2n\,(n-1)\binom{n-1}{d}\binom{n-2}{d-1}p^{2d}q^{2(n-d)-3} \\
&\qquad + (n-1)^2\binom{n-1}{d}\binom{n-2}{d-1}p^{2d+1}q^{2(n-d)-3} \\
&\qquad + n\binom{n-1}{d}p^d q^{n-1-d} - n\binom{n-1}{d}^2 p^{2d}q^{2(n-d)-2} \\
&= \left(n^2-n\right)\binom{n-2}{d-1}^2 p^{2d-1}q^{2(n-d)-3} - 2n\binom{n-1}{d}^2 p^{2d}q^{2(n-d)-3} \\
&\quad + p\left(n^2-n\right)\binom{n-1}{d}^2 p^{2d}q^{2(n-d)-3} + n\binom{n-1}{d}p^d q^{n-1-d} \\
&\qquad - nq\binom{n-1}{d}^2 p^{2d}q^{2(n-d)-3} \\
&\le n^2\binom{n-2}{d-1}^2 p^{2d-1}q^{2(n-d)-3} + n^2\binom{n-1}{d}^2 p^{2d+1}q^{2(n-d)-3} \\
&\qquad\qquad + n\binom{n-1}{d}p^d q^{n-1-d} =: a_n + b_n + c_n\,,
\end{aligned}
$$

where $a_n = n^2\binom{n-2}{d-1}^2 p^{2d-1}q^{2(n-d)-3}$, $b_n = n^2\binom{n-1}{d}^2 p^{2d+1}q^{2(n-d)-3}$ and $c_n = n\binom{n-1}{d}p^d q^{n-1-d}$.

Note that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{a_n}{n^2} = \sum_{n=1}^{\infty}\binom{n-2}{d-1}^2 p^{2d-1}q^{2(n-d)-3} &\le \sum_{n=d+1}^{\infty}\frac{n^{2(d-1)}}{[(d-1)!]^2}q^{2n-2d-3} \\
&\le C_1\sum_{n=1}^{\infty}n^{2d}q^{2n},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{b_n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n}\binom{n-1}{d}p^d q^{n-1-d} &\le \sum_{n=d+1}^{\infty}\left(\frac{1}{n}\right)\frac{n^d}{d!}p^d q^{n-1-d} \\
&\le C_2\sum_{n=1}^{\infty}n^d q^n,
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{c_n}{n^2} = \sum_{n=1}^{\infty}\binom{n-1}{d}^2 p^{2d+1}q^{2(n-d)-3} &\le \sum_{n=d+1}^{\infty}\frac{n^{2d}}{(d!)^2}p^{2d+1}q^{2(n-d)-3} \\
&\le C_3\sum_{n=1}^{\infty}n^{2d}q^{2n}\,,
\end{aligned}
$$

where C_1, C_2, C_3 are certain positive constants. By the ratio test, $\sum_{n=1}^{\infty}\frac{a_n}{n^2}$, $\sum_{n=1}^{\infty}\frac{b_n}{n^2}$ and $\sum_{n=1}^{\infty}\frac{c_n}{n^2}$ are finite which implies $\sum_{n=1}^{\infty}\frac{VarS_n}{n^2} < \infty$. Therefore by Proposition 2.1, we get the desired result.

Proof of Corollary 1.2. It follows directly from Theorem 1.1 in the case $d = 0$. □

Proof of Theorem 1.3. Let $D_{n,k} =: \{\vec{i} = (i_1, i_2, ..., i_k) : \ 1 \le i_1 < i_2 < ... < i_k \le n\}$ be the set of all possible combinations of k vertices. For each $\vec{i} \in D_{n,k}$, we define

$$X_{\vec{i}} = \begin{cases} 1, & \text{if there is an isolated tree in } G(n,p) \text{ which spanned} \\ & \text{by vertices } \vec{i} = (i_1, i_2, ..., i_k)\,; \\ 0, & \text{otherwise.} \end{cases}$$

Then all $X_{\vec{i}}$ s are not independent unless $k = 1$ and for $k \ge 2$,

$$EX_{\vec{i}} = P\left(X_{\vec{i}} = 1\right) = k^{k-2} p^{k-1} q^{k(n-k)+\binom{k}{2}-(k-1)} \tag{4}$$

(see [2], 137-143). Next, we set $S_{n,k} = \sum_{\vec{i} \in D_{n,k}} X_{\vec{i}}$. Clearly, $S_{n,k}$ is the number of isolated trees of order k in $G(n,p)$ and as in Barbour, Karoński and A. Rucinski [1],

$$Cov(X_{\vec{i}}, X_{\vec{j}}) = \begin{cases} EX_{\vec{i}} - \left(EX_{\vec{i}}\right)^2, & \text{if } \vec{i} = \vec{j}\,; \\ EX_{\vec{i}} EX_{\vec{j}} (q^{-k^2} - 1), & \text{if } \vec{i} \text{ and } \vec{j} \text{ have disjoint vertices}\,; \\ -EX_{\vec{i}} EX_{\vec{j}}, & \text{if } \vec{i} \ne \vec{j} \text{ and } \vec{i},\ \vec{j} \text{ have at least one vertex in common.} \end{cases}$$

For each $\vec{i} \in D_{n,k}$, we let:
$L_{\vec{i}} =: \{\vec{j} \in D_{n,k} : \ \vec{i} \text{ and} \vec{j} \text{ have disjoint vertices}\}$ and
$L'_{\vec{i}} =: \{\vec{j} \in D_{n,k} : \vec{i} \ne \vec{j} \text{ and } \vec{i}, \vec{j} \text{ have at least one vertex in common}\}$.
Hence, by (4),
$Cov(X_{\vec{i}}, X_{\vec{j}})$

$$= \begin{cases} k^{k-2} p^{k-1} q^{k(n-k)+\binom{k}{2}-k+1} \\ -k^{2(k-2)} p^{2k-2} q^{2k(n-k)+2\binom{k}{2}-2k+2}, & \text{if } \vec{j} = \vec{i}\,; \\ k^{2(k-2)} p^{2k-2} q^{2k(n-k)+2\binom{k}{2}-2k+2} \left(q^{-k^2} - 1\right), & \text{if } \vec{i} \in L_{\vec{i}}\,; \\ -k^{2(k-2)} p^{2k-2} q^{2k(n-k)+2\binom{k}{2}-2k+2}, & \text{if } \vec{j} \in L'_{\vec{i}}\,. \end{cases}$$

Then we obtain that

$$
\begin{aligned}
VarS_{n,k} &= \sum_{\vec{i}\in D_{n,k}} \sum_{\vec{j}\in D_{n,k}} Cov(X_{\vec{i}}, X_{\vec{j}}) \\
&= \sum_{\vec{i}\in D_{n,k}} \sum_{\vec{j}=\vec{i}} Cov(X_{\vec{i}}, X_{\vec{j}}) + \sum_{\vec{i}\in D_{n,k}} \sum_{\vec{j}\in L_{\vec{i}}} Cov(X_{\vec{i}}, X_{\vec{j}}) + \sum_{\vec{i}\in D_{n,k}} \sum_{\vec{j}\in L'_{\vec{i}}} Cov(X_{\vec{i}}, X_{\vec{j}}) \\
&= \tbinom{n}{k} [k^{k-2}p^{k-1}q^{k(n-k)+\binom{k}{2}-k+1} - k^{2(k-2)}p^{2k-2}q^{2k(n-k)+2\binom{k}{2}-2k+2}] \\
&\quad + \tbinom{n}{k}\binom{n-k}{k} [\, k^{2(k-2)}p^{2k-2}q^{2k(n-k)+2\binom{k}{2}-2k+2}(q^{-k^2}-1)\,] \\
&\quad - \tbinom{n}{k} \sum_{r=1}^{k-1} \tbinom{k}{r}\binom{n-k}{k-r} k^{2(k-2)}p^{2k-2}q^{2k(n-k)+2\binom{k}{2}-2k+2} \\
&\le \tbinom{n}{k} k^{k-2}p^{k-1}q^{k(n-k)+\binom{k}{2}-k+1} \\
&\quad + \tbinom{n}{k}\binom{n-k}{k} k^{2(k-2)}p^{2k-2}q^{2kn-2k^2-3k+2} \\
&=: c_1 \tbinom{n}{k} q^{kn} + c_2 \tbinom{n}{k}\binom{n-k}{k} q^{2kn} \qquad (5)
\end{aligned}
$$

where c_1 and c_2 are certain positive constants. Hence,

$$
\sum_{n=1}^{\infty} \frac{VarS_{n,k}}{\binom{n}{k}^2} \le c_1 \sum_{n=1}^{\infty} \frac{q^{kn}}{\binom{n}{k}} + c_2 \sum_{n=1}^{\infty} \frac{\binom{n-k}{k}}{\binom{n}{k}} q^{2kn}
\le c_1 \sum_{n=1}^{\infty} q^{kn} + c_2 \sum_{n=1}^{\infty} q^{2kn} < \infty\,,
$$

when we have used the fact that $0 < q < 1$ in the last inequality. Then the proof is complete. □

Proof of Theorem 1.4. Let H be a fixed connected graph consisting of k vertices and ℓ edges. For $k = 1$, the proof is complete by Corollary 1.2.

Suppose that $k \ge 2$.

Let $D_{n,k}$, $L_{\vec{i}}$ and $L'_{\vec{i}}$ be defined as in the proof of Theorem 1.3.

For each $\vec{i} \in D_{n,k}$, we define

$$
Y_{\vec{i}} = \begin{cases} 1, & \text{if subgraph spanned by vertices } \vec{i} = (i_1, i_2, ..., i_k) \\ & \text{ is an isolated copy of H;} \\ 0, & \text{otherwise.} \end{cases}
$$

Then all $Y_{\vec{i}}$ s are not independent. Note that there are $\frac{m!}{aut(H)}$ possible copies of H which spanned the vertices $\vec{i} = (i_1, i_2, ..., i_k)$, where $aut(H)$ stands for the number of automorphisms of H ([7], pp 141). Then we get

$$EY_{\vec{i}} = P\left(Y_{\vec{i}} = 1\right) = \frac{k!}{aut(H)} p^{\ell} q^{k(n-k)+\binom{k}{2}-\ell}$$

and $EY_{\vec{i}} = EY_{\vec{j}}$ for any $\vec{j},\ \vec{j} \in D_{n,k}$. Next, we set $S_{n,H} = \sum\limits_{\vec{i}\in D_{n,k}} Y_{\vec{i}}$. Clearly, $S_{n,H}$ is the number of isolated copies of H in $G(n,p)$.

If $\vec{i} = \vec{j}$, then $EY_{\vec{i}}Y_{\vec{j}} = EY_{\vec{i}}^2 = EY_{\vec{i}}$ and hence,

$$\begin{aligned} Cov(Y_{\vec{i}}, Y_{\vec{j}}) = EY_{\vec{i}}Y_{\vec{j}} - EY_{\vec{i}}EY_{\vec{j}} &= \frac{k!}{aut(H)} p^{\ell} q^{k(n-k)+\binom{k}{2}-\ell} \\ &\quad - (\frac{k!}{aut(H)})^2 p^{2\ell} q^{2k(n-k)+2\binom{k}{2}-2\ell} \\ &= \frac{k!}{aut(H)} p^{\ell} q^{kn-k^2+\binom{k}{2}-\ell} - (\frac{k!}{aut(H)})^2 p^{2\ell} q^{2kn-k^2-k-2\ell}. \quad (6) \end{aligned}$$

In case of $\vec{j} \in L'_{\vec{i}}$ we have $EY_{\vec{i}}Y_{\vec{j}} = 0$ which implies that

$$\begin{aligned} Cov(Y_{\vec{i}}, Y_{\vec{j}}) = EY_{\vec{i}}Y_{\vec{j}} - EY_{\vec{i}}EY_{\vec{j}} &= -(\frac{k!}{aut(H)})^2 p^{2\ell} q^{2k(n-k)+2\binom{k}{2}-2\ell} \\ &= -(\frac{k!}{aut(H)})^2 p^{2\ell} q^{2kn-k^2-k-2\ell} \quad (7) \end{aligned}$$

and in case of $\vec{j} \in L_{\vec{i}}$ we get

$$EY_{\vec{i}}EY_{\vec{j}} = P\left(Y_{\vec{i}} = 1, Y_{\vec{j}} = 1\right) = (\frac{k!}{aut(H)})^2 p^{2\ell} q^{2k(n-2k)+\binom{2k}{2}-2\ell}$$

and hence,

$$\begin{aligned} Cov(Y_{\vec{i}}, Y_{\vec{j}}) = EY_{\vec{i}}EY_{\vec{j}} - EY_{\vec{i}}EY_{\vec{j}} &= (\frac{k!}{aut(H)})^2 p^{2\ell} q^{2k(n-2k)+\binom{2k}{2}-2\ell} \\ - (\frac{k!}{aut(H)})^2 p^{2\ell} q^{2k(n-k)+2\binom{k}{2}-2\ell} &= (\frac{k!}{aut(H)})^2 p^{2\ell} q^{2kn-2k^2-k-2\ell} \\ &\quad - (\frac{k!}{aut(H)})^2 p^{2\ell} q^{2kn-k^2-k-2\ell}. \quad (8) \end{aligned}$$

From (6)-(8) we obtain that

$$\begin{aligned}
VarS_{n,H} &= \sum_{\vec{i}\in D_{n,k}} \sum_{\vec{j}\in D_{n,k}} Cov(Y_{\vec{i}}, Y_{\vec{j}}) \\
&= \sum_{\vec{i}\in D_{n,k}} \sum_{\vec{j}=\vec{i}} Cov(Y_{\vec{i}}, Y_{\vec{j}}) + \sum_{\vec{i}\in D_{n,k}} \sum_{\vec{j}\in L_{\vec{i}}} Cov(Y_{\vec{i}}, Y_{\vec{j}}) + \sum_{\vec{i}\in D_{n,k}} \sum_{\vec{j}\in L'_{\vec{i}}} Cov(Y_{\vec{i}}, Y_{\vec{j}}) \\
&= \binom{n}{k} \Big[\frac{k!}{aut(H)} p^\ell q^{kn-k^2+\binom{k}{2}-\ell} - \Big(\frac{k!}{aut(H)}\Big)^2 p^{2\ell} q^{2kn-k^2-k-2\ell} \Big] \\
&\quad + \binom{n}{k}\binom{n-k}{k} \Big[\Big(\frac{k!}{aut(H)}\Big)^2 p^{2\ell} q^{2kn-2k^2-k-2\ell} \\
&\quad - \Big(\frac{k!}{aut(H)}\Big)^2 p^{2\ell} q^{2kn-k^2-k-2\ell} \Big] - \binom{n}{k} \sum_{r=1}^{k-1} \binom{k}{r}\binom{n-k}{k-r} \Big(\frac{k!}{aut(H)}\Big)^2 p^{2\ell} q^{2kn-k^2-k-2\ell} \\
&\le \binom{n}{k} \frac{k!}{aut(H)} p^\ell q^{kn-k^2+\binom{k}{2}-\ell} + \binom{n}{k}\binom{n-k}{k} \Big(\frac{k!}{aut(H)}\Big)^2 p^{2\ell} q^{2kn-2k^2-k-2\ell} \\
&=: c_{H_1} \binom{n}{k} q^{kn} + c_{H_2} \binom{n}{k}\binom{n-k}{k} q^{2kn}, \qquad (9)
\end{aligned}$$

where c_{H_1} and c_{H_2} are unimportant positive constants. Hence,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{VarS_{n,H}}{\binom{n}{k}^2} &\le c_{H_1} \sum_{n=1}^{\infty} \frac{q^{kn}}{\binom{n}{k}} + c_{H_2} \sum_{n=1}^{\infty} \frac{\binom{n-k}{k}}{\binom{n}{k}} q^{2kn} \\
&\le c_{H_1} \sum_{n=1}^{\infty} q^{kn} + c_{H_2} \sum_{n=1}^{\infty} q^{2kn} < \infty. \qquad (10)
\end{aligned}$$

Then the proof is complete. □

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