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IMPROVED GREEDY ALGORITHM FOR MAXIMUM COVERAGE PROBLEM WITH GROUP BUDGET CONSTRAINTS

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Abstract: The main problem considered is maximum coverage problem with group budget constraints, which generalizes the budgeted maximum coverage problem. This problem is NP-hard and no exact approximation results are known for it. The contribution of this paper is a $(1 - e^{-1})$-approximation algorithm for the maximum coverage problem with group budget constraint. We also show that this approximation factor is the best possible, unless $P = NP$.

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1. Introduction

We consider the following coverage problem: A collection of sets $S = \{S_1, S_2, \cdots S_m\}$ with associated costs $c(S_j), j = 1, 2, \cdots m$ is defined over a ground set $X = \{x_1, x_2, \cdots, x_n\}$ with associated weights $w(x_i), i = 1, 2, \cdots, n$. We also give sets $G_1, G_2, \cdots, G_l$, each set being a subset of S. We call G_k a group. By

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making copies of sets, if necessary, we can assume that the groups G_k are disjoint from each other. Further, we give a budget B_k for group $G_k, k = 1, 2, \cdots, l$ and an overall budget B. The goal is to find a collection of sets $S' \subseteq S$, such that the total cost of elements in S' is at most B. Further for any group G_k, the total cast of elements in $S' \bigcap G_k$ can be at most B_k, and the total weight of elements covered by S' is maximized. We call this problem is the maximum coverage problem with group budget constraints (MCG). It can be formulated by the following program

$$(MCG) \qquad \max_{S' \subseteq S} \{w(S') : c(S' \cap G_k) \leq B_k; c(S') \leq B\}.$$

Here, $w(S')$ denote the total weight of elements covered by S'.

The budgeted maximum coverage problem, where no group budget constraints, has been considered in [5]. Many applications arising in circuit layout, job scheduling, facility location, and other areas, may be modeled using the maximum coverage problem [3].

The maximum coverage problem with group budget constraints introduces a more flexible model for the applications mentioned above. Furthermore, the problem also can be used to solve the k-traveling repairmen problem [4] and the orienteering problem with time windows. Those applications have been discussed in [1] by Chekuri and Kumar. They seem to be first to consider the MCG and present an $6(\alpha + 1)-$ approximation algorithm for the MCG by value-giving oracle. There, α is approximation factor for the given oracle.

In this paper we present a $(1 - e^{-1})-$ approximation algorithm for MCG. The performance guarantee (we say that an algorithm has performance guarantee $\alpha < 1$ if it always delivers a solution of value at least α times of the value of an optimal solution) of the approximation algorithm we present in this paper improves on that of the known $6(\alpha + 1)-$ approximation algorithm due to Chekuri and Kumar. We also show that the greedy algorithm combined with the partial enumeration procedure has performance guarantee $1 - e^{-1}$ and this is the best possible performance guarantee achievable in polynomial time unless $P = NP$, see [2].

2. The Modified Greedy Algorithm for MCG

In this section we show that the modified greedy algorithm gives constant factor approximate ratios for MCG. First we consider the special case of the problem, which there are no group budget constraints on the collection of sets S. Krause et al [6] propose a partial enumeration heuristic which enumerates all subsets of up to d elements for some constant $d > 0$, and complements these subsets using the modified greedy algorithm. They prove that this algorithm guarantees a $(1-e^{-1})-$ approximate algorithm for the maximizing a submodular set function with budget constraints. The algorithm for special case of the MCG is as follows:

Algorithm 1. Approximation algorithm for special case of MCG

$A(S, B, d)$

Input : $d > 0, B > 0, S$

Output : *Selection* $S^{'} \subseteq S$

begin :

$T_1 \leftarrow \arg\max\{w(H) : H \subseteq S, |H| < d, c(H) \leq B\};$

$T_2 \leftarrow \emptyset; U \leftarrow \emptyset$

foreach $H \subseteq S, |H| = d, c(H) \leq B$ **do**

$U \leftarrow S \backslash H;$

while $U \neq \emptyset$ **do**

foreach $S_i \in U$ **do** $\triangle S_i \leftarrow w(H \bigcup S_i) - w(H);$

$i^* \leftarrow \arg\max\{\triangle S_i / c_i : \ S_i \in S\};$

if $c(H) + c_{i^*} \leq B$ **then** $H \leftarrow H \bigcup S_i$

end

if $w(H) > w(T_2)$ **then** $T_2 \leftarrow H$

end

if $w(T_1) > w(T_2)$, **output** T_1, **otherwise**, **output** T_2

end

Theorem 1. *For* $d > 3$, *Algorithm* 1 *achieves an approximation factor of* $(1 - e^{-1})$ *for the budgeted maximum coverage problem.*

The proof of Theorem 1 is omitted. For detail information you can see [5].

We denote Algorithm 1 by $A(S, B, d)$, which means input a collection of set S, a budget constant B and some constant $d > 0$. $A(S, B, d)$ output a set $S^{'} \subseteq S$ such that the value of $w(S^{'})$ at least $1 - e^{-1}$ times of the value of an optimal solution.

We now consider the MCG. We give a greedy algorithm for this problem which is similar in spirit to the one for the special case but differs in some technical details. In this case we should find a collection of sets $S^{'} \subseteq S$ that maximizing the total weight of the elements covered by $S^{'}$. At the same time, $c(S^{'}) \leq B$ and $c(S^{'} \bigcap G_k) \leq B_k, k = 1, 2, \cdots, m$. We assume without loss of generality that $B \leq \sum_{k=1}^{m} B_k$. So if we use Algorithm1 to each group and delivers a solution of value at least $1 - e^{-1}$ times of the value of an optimal solution for the group. Then the total cost of the collection obtained by Algorithm 1 in each group may violate the overall cost bound B. Now we show how to modify the cost bound for each group to respect the overall cost bound. Let $B_k^{'} = \min\{B - \sum_{l=0}^{k-1} B_l, B_k\}$ and $B_0 = 0$. Then we use Algorithm1 to each group G_k with cost bound $B_k^{'}$ and output a collection of sets $T_k \subseteq G_k$. The algorithm for MCG is described in more detail below.

Algorithm 2. Approximation algorithm for MCG

Input: $d > 0$, $m > 0$, $B > 0$, $B_0 = 0$, $B_1 > 0, \cdots, B_m > 0$, S, G_1, $\cdots$, G_m

Output : $S^{'} \subseteq S$

begin

  $G^{'} \leftarrow G_k,\ B^{'} \leftarrow \min\{B - \sum_{l=0}^{k} B_l, B_k\},\ S^{'} \leftarrow \emptyset$

    $k = 1$

    if $k \leq m$ **then**

      call $A(G^{'}, B^{'}, d)$

      $T_k \leftarrow A(G^{'}, B^{'}, d)$

      $S^{'} \leftarrow S^{'} \bigcup T_k$

      $k = k + 1$

    end

  Output $S^{'}$

end

Next, we prove the following theorem about the performance guarantee of Theorem 2.

Theorem 2. *For $d > 3$, Algorithm 2 achieves an approximation factor of $(1 - e^{-1})$ for the MCG.*

Proof. Let OPT denote the collection of sets in an optimal solution of MCG. Let $OPT_k(k = 1, 2, \cdots, m)$ denote the collection of sets in an optimal solution of each group. Clearly, $w(OPT_k) \geq w(OPT \bigcap G_k), k = 1, 2, \cdots, m$. From Algorithm 1, we have that $w(T_k) \geq (1 - e^{-1})w(OPT_k)$ and

$$w(S^{'}) = w(T_1 \bigcup T_2 \bigcup \cdots \bigcup T_m) = w(T_1) + w(T_2) + \cdots + w(T_m)$$

$$\geq (1-e^{-1})[w(OPT_1) + w(OPT_2) + \cdots + w(OPT_m)]$$
$$\geq (1-e^{-1})[w(OPT \bigcap G_1) + w(OPT \bigcap G_2) + \cdots + w(OPT \bigcap G_m)]$$
$$= (1-e^{-1})w(OPT)\,.$$

Hence $w(S') \geq (1-e^{-1})w(OPT)$, Theorem 2 holds true. □

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