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SEMI ω^0-CONTINUOUS FUNCTIONS

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Abstract: In this paper, we introduce and study the class of semi ω^0-open subsets of a topological space (X, τ) then we study the semi ω^0-interior and the semi ω^0-closure. Also we define what we call semi ω^0-continuous function and we give several properties and characterizations of this function.

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1. Introduction

Let (X, τ) be a topological space and A be a subset of X. We denote the closure of A (the interior of A respectively) by $cl(A)$ ($int(A)$ respectively). A is called $\omega^0 - open$ (see [1]) if for every $x \in A$, there exists an open subset $U_x \subseteq X$ containing x such that $U_x - int(A)$ is countable. The complement of an ω^0-open set is called ω^0-closed . The family of all ω^0-open subsets of (X, τ) is denoted by $\omega^0 O(X, \tau)$. It is well known that $\omega^0 O(X, \tau)$ forms a topology on X which we denote by τ_{ω^0}.

We always use R and Q to denote the set of real numbers and the set of rational numbers respectively. Also we use τ_s and τ_{disc} to denote the standard and discrete topologies, respectively.

In Section 2, we introduce semi ω^0-open subsets of a topological space (X, τ) and we study their properties and characterizations. In Section 3, we study the semi ω^0-interior and semi ω^0-closure . Analogous to [3, 4, 5, 6], in Section 4, we define the class of semi ω^0-continuous functions and establish some properties

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of this new class. Finally, in Section 5, we define and study the concepts of $\omega^0 - irresolute$ and semi $\omega^0 - irresolute$ functions.

2. Semi ω^0-Open Subsets

If A is a subset of a space (X, τ), then $\omega^0 int(A)$ and $\omega^0 cl(A)$ denotes the interior of A and the closure of A, respectively, in the space (X, τ_{ω^0}).

Definition 2.1. A subset A of a space (X, τ) is called semi ω^0-open if $A \subseteq cl(\omega^0 int(A))$.

The family of all semi ω^0-open subsets of (X, τ) will be denoted by $S\omega^0 O(X, \tau)$.

Theorem 2.2. *A subset A of a space (X, τ) is* semi ω^0-open *if and only if there exists an* ω^0-open *subset U of (X, τ) such that $U \subseteq A \subseteq cl(U)$.*

Proof. $\Longrightarrow$) let $A \in S\omega^0 O(X, \tau)$. Then $A \subseteq cl(\omega^0 int(A))$. Put $U = \omega^0 int(A)$, then $U \subseteq A \subseteq cl(U)$.

$\Longleftarrow$) Let A be a subset of (X, τ) and assume that there exists an ω^0-open subset U of (X, τ) such that $U \subseteq A \subseteq cl(U)$. But $U = \omega^0 int(U) \subseteq \omega^0 int(A)$ and thus $cl(U) \subseteq cl(\omega^0 int(A))$. Hence $A \subseteq cl(U) \subseteq cl(\omega^0 int(A))$. □

Clearly every open set is ω^0-open and every ω^0-open set is semi ω^0-open.

Theorem 2.3. *Any arbitrary union of* semi ω^0-open *subsets of a space (X, τ) is* semi ω^0-open .

Proof. Let $\{A_\alpha : \alpha \in \Delta\}$ be a collection of *semi* ω^0-open subsets of (X, τ). For each $\alpha \in \Delta$, there exists $U_\alpha \in \omega^0 O(X, \tau)$ such that $U_\alpha \subseteq A_\alpha \subseteq cl(U_\alpha)$. Now $\cup_{\alpha\in\Delta} U_\alpha \subseteq \cup_{\alpha\in\Delta} A_\alpha \subseteq \cup_{\alpha\in\Delta} cl(U_\alpha) \subseteq cl(\cup_{\alpha\in\Delta} U_\alpha)$ and $\cup_{\alpha\in\Delta} U_\alpha \in \omega^0 O(X, \tau)$. Then $\cup_{\alpha\in\Delta} A_\alpha \in S\omega^0 O(X, \tau)$. □

The intersection of two semi ω^0-open sets need not to be *semi* ω^0-open.

Example 2.4. Let $X = R$ and $\tau = \{U \subseteq R : 1 \in U$ and $R - U$ is finite $\} \cup \{U \subseteq R : 1 \notin u\}$. Put $A = Q$ and $B = (R - Q) \cup \{1\}$. Then $A, B \in S\omega^0 O(X, \tau)$ while $A \cap B = \{1\} \notin S\omega^0 O(X, \tau)$, since $\omega^0 int(\{1\}) = \phi$.

Theorem 2.5. *The intersection of a* semi ω^0-open *set and an open set is* semi ω^0-open .

Proof. Let $A \in S\omega^0 O(X, \tau)$ and V be an open set in the space (X, τ). Then there exists $U \in \omega^0 O(X, \tau)$ such that $U \subseteq A \subseteq cl(U)$. Now $U \cap V \subseteq A \cap V \subseteq cl(U) \cap V \subseteq cl(U \cap V)$, where $U \cap V \in \omega^0 O(X, \tau)$. Then $A \cap V \in S\omega^0 O(X, \tau)$. □

Definition 2.6. A subset F of a space (X, τ) is called semi ω^0-closed if $int(\omega^0 cl(F)) \subseteq F$.

The family of all semi ω^0-closed subsets of a space (X, τ) will be denoted by $S\omega^0C(X, \tau)$.

Clearly every ω^0-closed set is semi ω^0-closed

The following two theorems are dual to Theorem 2.3 and Theorem 2.5 respectively and their proofs are omitted.

Theorem 2.7. *The arbitrary intersection of* semi ω^0-closed *subsets of a space* (X, τ) *is semi* ω^0-closed .

Theorem 2.8. *The union of a* semi ω^0-closed *set and a closed set is* semi ω^0-closed .

3. Semi ω^0-Interior and Semi ω^0-Closure

Now we define the concepts of semi ω^0-interior and semi ω^0-closure.

Definition 3.1. If A is a subset of a space (X, τ), then the union of all semi ω^0-open subsets of (X, τ) contained in A is called the semi ω^0-interior of A, denoted by $s\omega^0 int(A)$.

By Theorem 2.3 we observe that $s\omega^0 int(A) \in S\omega^0 O(X, \tau)$.

Theorem 3.2. *If A is a subset of a space (X, τ), then $s\omega^0 int(A) = A \cap cl(\omega^0 int(A))$.*

Proof. Since $s\omega^0 int(A) \in S\omega^0 O(X, \tau)$, we have $s\omega^0 int(A) \subseteq cl(\omega^0\ int(s\omega^0 int(A))) \subseteq cl(\omega^0 int(A))$. But $s\omega^0 int(A) \subseteq A$, so $s\omega^0 int(A) \subseteq A \cap cl(\omega^0 int(A))$. Now, notice that $A \cap cl(\omega^0 int(A)) \subseteq cl(\omega^0 int(A)) \subseteq cl(\omega^0 int(A \cap \omega^0 int(A))) \subseteq cl(\omega^0 int(A \cap cl(\omega^0 int(A))))$. Then $A \cap cl(\omega^0 int(A))$ is a semi ω^0-open set contained in A, therefore $A \cap cl(\omega^0 int(A)) \subseteq s\omega^0 int(A)$. □

Then $s\omega^0 int(A) = A \cap cl(\omega^0 int(A))$.

Corollary 3.3. *Let A be a subset of a space (X, τ), then $s\omega^0 int(A) = A$ if and only if $A \in S\omega^0 O(X, \tau)$.*

Definition 3.4. If A is a subset of a space (X, τ) then the intersection of all semi ω^0-closed subsets of (X, τ) containing A is called the semi ω^0-closure of A, denoted by $s\omega^0 cl(A)$.

By Theorem 2.7 we observe that $s\omega^0 cl(A) \in S\omega^0 C(X, \tau)$.

Theorem 3.5. *If A is a subset of a space (X, τ), then $s\omega^0 cl(A) = A \cup int(\omega^0 cl(A))$.*

Proof. Since $s\omega^0 cl(A) \in S\omega^0 C(X, \tau)$ we have $int(\omega^0 cl(s\omega^0 cl(A))) \subseteq s\omega^0 cl(A)$. Then $int(\omega^0 cl(A)) \subseteq int(\omega^0 cl(s\omega^0 cl(A))) \subseteq s\omega^0 cl(A)$ and so $A \cup int(\omega^0 cl(A)) \subseteq$

$s\omega^0 cl(A)$.

On the other hand, notice that $int(\omega^0 cl(A \cup int(\omega^0 cl(A)))) \subseteq int(\omega^0 cl(A \cup \omega^0 cl(A))) = int(\omega^0 cl(A)) \subseteq A \cup int(\omega^0 cl(A))$. Then $A \cup int(\omega^0 cl(A))$ is a semi ω^0-closed subset of (X, τ) containing A and so $s\omega^0 cl(A) \subseteq A \cup int(\omega^0 cl(A))$. Thus $s\omega^0 cl(A) = A \cup int(\omega^0 cl(A))$. □

Corollary 3.6. *Let A be a subset of a space (X, τ) then $s\omega^0 cl(A) = A$ if and only if $A \in S\omega^0 C(X, \tau)$.*

Corollary 3.7. *Let A be a subset of a space (X, τ) then:*

a) $s\omega^0 int(X - A) = X - s\omega^0 cl(A)$.

b) $s\omega^0 cl(X - A) = X - s\omega^0 int(A)$.

4. Semi ω^0-Continuous Functions

A function $f : X \to Y$ is ω^0-continuous (see [2]), if the inverse image of every open subset of Y is ω^0-open in X.

Definition 4.1. A function $f : (X, \tau) \to (Y, \tau')$ is called semi ω^0-continuous at $x \in X$ if for every $V \in \tau'$ containing $f(x)$, there exists $U \in S\omega^0 O(X, \tau)$ containing x such that $f(U) \subseteq V$. f is semi ω^0-continuous if it is semi ω^0-continuous at every $x \in X$.

It is obvious that every ω^0-continuous function is semi ω^0-continuous. The following example shows that the converse need not to be true.

Example 4.2. Let $X = R$ and $\tau = \{U \subseteq R : 1 \in U$ and $R - U$ is finite $\} \cup \{U \subseteq R : 1 \notin U\}$ and let $f : (R, \tau) \to (\{0, 1\}, \tau_{disc})$ be the function defined by $f(x) = 0$ if $x \in Q$ and $f(x) = 1$ if $x \in R - Q$ then f is semi ω^0-continuous but not ω^0-continuous since $f^{-1}(\{0\}) = Q \notin \omega^0 O(X, \tau)$.

The following result is now clear.

Theorem 4.3. *A function $f : (X, \tau) \to (Y, \tau')$ is* semi ω^0-continuous *if and only if for every $V \in \tau'$, $f^{-1}(V) \in S\omega^0 O(X, \tau)$.*

The easy proof of the following theorem is omitted.

Theorem 4.4. *If $f : (X, \tau) \to (Y, \tau')$ is* semi ω^0-continuous *and $g : (Y, \tau') \to (Z, \tau'')$ is continuous then $gof : (X, \tau) \to (Z, \tau'')$ is* semi ω^0-continuous .

Notice that the composition of two semi ω^0-continuous functions need not to be a semi ω^0-continuous function as it is shown in the next example.

Example 4.5. Let $Y = \{0, 1\}$ and $\tau = \{\phi, Y, \{0\}\}$ and let $f : (R, \tau_s) \to$

(Y, τ) be the function defined by $f(x) = 1$ if $x \in Q$ and $f(x) = 0$ if $x \in R - Q$ and $g : (Y, \tau) \to (Y, \tau)$ be defined as follows: $g(0) = 1$ and $g(1) = 0$. Then f and g are semi ω^0-continuous but gof is not semi ω^0-continuous since $(gof)^{-1}(\{0\}) = f^{-1}(\{1\}) = Q \notin S\omega^0O(R, \tau_s)$.

Theorem 4.6. *Let $f : (X, \tau) \to (Y, \tau')$ be a continuous, open and onto function. If $U \in S\omega^0O(X, \tau)$ then $f(U) \in S\omega^0O(Y, \tau')$.*

Proof. Let $U \in S\omega^0O(X, \tau)$. Then $U \subseteq cl(\omega^0 int(U))$ and since f is continuous we have $f(U) \subseteq f(cl(\omega^0 int(U))) \subseteq cl(f(\omega^0 int(U)))$. Now f is open and onto, therefore $f(\omega^0 int(U)) \subseteq \omega^0 int(f(U))$. Then $f(U) \subseteq cl(\omega^0 int(f(U)))$. So $f(U) \in S\omega^0O(Y, \tau')$. □

In a similar way we can prove the following theorem.

Theorem 4.7. *Let $f : (X, \tau) \to (Y, \tau')$ be a continuous, open and onto function. If $U \in S\omega^0O(Y, \tau')$, then $f^{-1}(U) \in S\omega^0O(X, \tau)$.*

Theorem 4.8. *Let $f : (X, \tau) \to (Y, \tau')$ be a continuous, open and onto function. Then $g : (Y, \tau') \to (Z, \tau'')$ is* semi ω^0-continuous *if and only if $gof : (X, \tau) \to (Z, \tau'')$ is* semi ω^0-continuous .

Proof. $\Longrightarrow$) Let g be semi ω^0-continuous and $V \in \tau''$. Then $g^{-1}(V) \in S\omega^0O(Y, \tau')$. By Theorem 4, we have $f^{-1}(g^{-1}(V)) \in S\omega^0O(X, \tau)$. So $(gof)^{-1}(V) \in S\omega^0O(X, \tau)$.

$\Longleftarrow$) Let gof be semi ω^0-continuous and $V \in \tau''$ then $(gof)^{-1}(V) \in S\omega^0O(X, \tau)$. By Theorem 3, we have $f((gof)^{-1}(V)) \in S\omega^0O(Y, \tau')$. So $g^{-1}(V) = f(f^{-1}(g^{-1}(V))) \in S\omega^0O(Y, \tau')$. □

5. ω^0-Irresolute and Semi ω^0-Irresolute Functions

Definition 5.1. A function $f : (X, \tau) \to (Y, \tau')$ is called ω^0-irresolute if $f^{-1}(V) \in \omega^0O(X, \tau)$, for every $V \in \omega^0O(Y, \tau')$.

Definition 5.2. A function $f : (X, \tau) \to (Y, \tau')$ is called semi ω^0-irresolute if $f^{-1}(V) \in S\omega^0O(X, \tau)$, for every $V \in S\omega^0O(Y, \tau')$.

Notice that every ω^0-irresolute function is ω^0-continuous and so it is semi ω^0-continuous.

It is clear that a function $f : (X, \tau) \to (Y, \tau')$ is ω^0-irresolute if and only if $f : (X, \tau_{\omega^0}) \to (Y, \tau'_{\omega^0})$ is continuous. Therefore the following theorem can be easily proved.

Theorem 5.3. *The following conditions are equivalent for a function f :*

$(X,\tau) \to (Y,\tau')$:

a) f is ω^0-irresolute .

b) $f^{-1}(C)$ is an ω^0-closed *subset of X, for every* ω^0-closed *subset C of Y.*

c) $f(\omega^0 cl(A)) \subseteq \omega^0 cl(f(A))$, for every subset A of X.

d) $\omega^0 cl(f^{-1}(B)) \subseteq f^{-1}(\omega^0 cl(B))$, for every subset B of Y.

Example 5.4. Let $X = R$ with the standard topology τ_s and $Y = \{0,1\}$ with the topology $\tau' = \{\phi, Y, \{0\}\}$ and let $f : (R,\tau_s) \to (Y,\tau')$ be the function defined by $f(x) = 0$ if $x \in Q$ and $f(x) = 1$ if $x \in R - Q$ then f is semi ω^0-continuous but it is neither ω^0-irresolute nor semi ω^0-irresolute .

The proof of the following theorem is straightforward.

Theorem 5.5. *If $f : (X,\tau) \to (Y,\tau')$ is* semi ω^0-irresolute *(*ω^0-irresolute *, respectively) and $g : (Y,\tau') \to (Z,\tau'')$ is* semi ω^0-continuous *(*ω^0-continuous *, respectively) then gof is* semi ω^0-continuous *(*ω^0-continuous , *respectively).*

Theorem 5.6. *Let $f : (X,\tau) \to (Y,\tau')$ be an open function. If f is* ω^0-irresolute *then f is* semi ω^0-irresolute .

Proof. Let $V \in S\omega^0 O(Y,\tau')$. Then there exists $U \in \omega^0 O(Y,\tau')$ such that $U \subseteq V \subseteq cl(U)$. Now $f^{-1}(U) \in \omega^0 O(X,\tau)$ since f is ω^0-irresolute . But f is an open function, so $f^{-1}(cl(U)) \subseteq cl(\ f^{-1}(U))$. Then $f^{-1}(U) \subseteq f^{-1}(V) \subseteq f^{-1}(cl(U)) \subseteq cl(f^{-1}(U))$, so $f^{-1}(V) \in S\omega^0 O(X,\tau)$.

Observe that the function f defined in Example 4.2. is open and semi ω^0-irresolute but not ω^0-irresolute , so the converse of the last theorem need not to be true.

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