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ALMOST CONTACT METRIC STRUCTURES ON HYPERSURFACES OF ALMOST HERMITIAN MANIFOLDS

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Abstract: The Cartan structural equations of an arbitrary almost contact metric structure induced on an oriented hypersurface of an almost Hermitian manifold are obtained.

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1. Introduction

The theory of almost contact metric structures occupies one of the leading places in modern differential-geometrical researches. It is due to a number of its applications in mathematical physics (for example, in classical mechanics [1] and in theory of geometrical quantization [8]). Furthermore, we mark out the riches of the internal contents of the theory of almost contact metric structures

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as well as the close connection of this theory with other sections of geometry, see [11, 16].

We recall that an almost contact metric structure on an odd-dimensional manifold N is defined by the system of tensor fields $\{\Phi,\ \xi,\ \eta,\ g\}$ on this manifold, where ξ is a vector, η is a covector, Φ is a tensor of the type $(1,1)$ and $g = \langle .,. \rangle$ is the Riemannian metric [6]. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1,\ \Phi(\xi) = 0,\ \eta \circ \Phi = 0,\ \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X\ , \Phi Y \rangle = \langle X\ , Y \rangle - \eta(X)\eta(Y),\ X, Y \in \aleph(N),$$

where $\aleph(N)$ is the module of smooth vector fields on N. As an example of an almost contact metric structure we can consider the cosymplectic structure [3, 4, 6, 10], that is characterized by the following condition:

$$\nabla\eta = 0, \quad \nabla\Phi = 0,$$

where ∇ is the Levi-Civita connection of the metric. It has been proved that the manifold, admitting the cosymplectic structure, is locally equivalent to a product $M \times R$, where is M a Kählerian manifold, see [11].

The almost contact metric structures are closely connected to the almost Hermitian structures. For instance, if $(N, \{\Phi,\ \xi,\ \eta,\ g\})$ is an almost contact metric manifold, then an almost Hermitian structure is induced on the product $N \times R$, see [10]. If this almost Hermitian structure is integrable, then the input almost contact metric structure is called normal. As it is known, a normal contact metric structure is called Sasakian, see [6]. On the other hand, we can characterize the Sasakian structure by the following condition:

$$\nabla_X(\Phi)Y = \langle X, Y \rangle\, \xi - \eta(Y)X,\ X, Y \in \aleph(N). \tag{1}$$

For example, Sasakian structures are induced on totally umbilical hypersurfaces in a Kählerian manifold, see [5, 6]. As it is well known, the Sasakian structures have many remarkable properties and play a fundamental role in contact geometry.

In 1972 Katsuei Kenmotsu has introduced a new class of almost contact metric structures [9], defined by the condition

$$\nabla_X(\Phi)Y = \langle \Phi X, Y \rangle\, \xi - \eta(Y)\Phi X,\ \ X, Y \in \aleph(N). \tag{2}$$

The Kenmotsu manifolds are normal and integrable, but they are not contact, consequently, they can not be Sasakian. In spite of the fact that the conditions (1) and (2) are similar, the properties of Kenmotsu manifolds are to some extent antipodal to the Sasakian manifolds properties, see [10]. Note that the new investigation [10] in this field contains a detailed description of Kenmotsu manifolds as well as a collection of examples of such manifolds.

2. Preliminaries

We consider an almost Hermitian manifold, i.e. a $2n$-dimensional manifold $\left(M^{2n}, J, g = \langle\ \cdot, \cdot\rangle\right)$, having a Riemannian metric $g = \langle\ \cdot,\ \cdot\rangle$ and an almost complex structure J. Besides the following condition must hold

$$\langle JX,\ JY\rangle = \langle X,\ Y\rangle\,,\quad \forall X,\ Y \in \aleph(M^{2n}),$$

where $\aleph(M)$ is the module of smooth vector fields on M^{2n}, see [7]. All manifolds, tensor fields and similar objects are assumed to be of the class C^∞.

Let $N \subset M^{2n}$ be an oriented hypersurface of the considered almost Hermitian manifold. We denote as $\aleph_N(M)$ the reduction of the module of vector fields $\aleph(M)$ onto N:

$$\aleph_N(M) = \bigcup_{p\in N} T_p(M).$$

It is evident that

$$\aleph(N) \subset \aleph_N(M).$$

Let $\aleph^\perp(N)$ be the orthogonal supplement of $\aleph(N)$ in $\aleph_N(M)$. Obviously, $\aleph^\perp(N)$ is a one-dimensional submodule of $\aleph_N(M)$. If $p \in N$ is an arbitrary point, then

$$T_p^\perp(N) = \{X \in T_p(M)\,|\langle X, Y\rangle = 0; Y \in T_p(N)\}\,,\ \dim\left(T_p^\perp(N)\right) = 1.$$

Consequently, the unit vector at the point p

$$n_0 \in T_p(M)$$

can be considered as a basis of $T_p^\perp(N)$, and that is why

$$\{(n_0)_p\,|p \in N\} \in \aleph_N(M)$$

is a basis of $\aleph^\perp(N)$.

Introducing the notation $\xi = J(n_0)$, we have $\xi \in \aleph(N)$, thus

$$\langle \xi, n_0\rangle = \langle J(n_0), n_0\rangle = -\Omega(n_0, n_0) = 0,$$

so $\xi \perp n_0$, and according to [16], $\xi \in \aleph(N)$.

Let us denote:

$$\eta \in \aleph_N^*(M),\ \eta(X) = \langle \xi, X\rangle\ \text{ and }\ \zeta \in \aleph_N^*(M),\ \zeta(X) = \langle n_0, X\rangle\,.$$

Then we obtain:

$$\eta(X) = \langle \xi, X \rangle = \langle J(n_0), X \rangle = -\langle n_0, JX \rangle = -\zeta \circ J(X),$$

i.e. $\eta = -\zeta \circ J$. So, there are some projectors in the module of vector fields $\aleph_N(M)$. Namely:

$\overline{n}_1 = \zeta \otimes n_0$ is the projector onto normal vector;

$\Pi_1 = id - \overline{n}_1$ is the projector onto submodule $\aleph(N)$;

$\overline{n}_2 = \eta \otimes \xi$;

$\Pi_2 = id - \overline{n}_2$.

Let us denote $\mathrm{Im}(\overline{n}_2) = \Psi$. Then we can consider some new projectors:

$$\overline{n}_3 = \overline{n}_1 + \overline{n}_2;$$

$$\Pi_3 = id - \overline{n}_3.$$

Remark that the images of the projectors $\overline{n}_3$ and Π_3 are orthogonal, and that is why

$$\aleph_N(M) = \mathrm{Im}(n_3) + \mathrm{Im}(\Pi_3).$$

Let us denote $(\Pi_3) = \Lambda$. We get that

$$\Psi = \Lambda(\xi) \subset \aleph(N),$$

where $\aleph(N) = \Lambda \oplus \Psi$ is a direct orthogonal sum. In fact,

$$\begin{aligned}\aleph_N(M) = \mathrm{Im}(n_3) + \mathrm{Im}(\Pi_3) = \Lambda \oplus \Lambda(\xi, n_0) = \Lambda \oplus \Lambda(\xi) \oplus \Lambda(n_0)\\ = \Lambda \oplus \Psi \oplus \Lambda(n_0),\end{aligned}$$

so

$$\Lambda \oplus \Psi = \Lambda^{\perp}(n_0) = \aleph(N).$$

Taking into account that $\Lambda(\xi, n_0)$ is invariant with respect to J, we can conclude that the orthogonal supplement Λ is also invariant with respect to J. We determine the endomorphism Φ of the module $\aleph_N(M)$ by the following formula:

$$\Phi = J \circ \Pi_3.$$

As it is evident, if Λ and Ψ are invariant with respect to J and Π_3, then this objects are invariant with respect to Φ.

By direct computing of $\Phi^2(X),\ X \in \aleph_N(M)$, we get:

$$\Phi^2 = -id + \zeta \otimes n_0 + \eta \otimes \xi.$$

In particular, if $X \in \aleph(N)$, then

$$\zeta(X) = \langle n_0, X \rangle = 0.$$

So,

$$\Phi^2(X) = -X + \eta(X)\xi, X \in \aleph(N).$$

We have obtained that the operator Φ on $\aleph(N)$ complies with the following condition:

$$\Phi^2 = -id + \eta \otimes \xi.$$

As a result we have that the following tensors are defined on $\aleph(N)$:

$$\xi,\ \eta,\ \Phi,\ g,$$

where

$$\xi = J(n_0),\ \eta(X) = \langle \xi, X \rangle,\ \Phi = J \circ \Pi_3 \big|_{\aleph(N)},\ g = \langle\ \cdot, \cdot \rangle \big|_{\aleph(N)}.$$

Moreover, some conditions are fulfilled:

1) $\eta(\xi) = \langle \xi, \xi \rangle = 1$;

2) $\eta \circ \Phi = \eta \circ J \circ \Pi_3 \big|_{\aleph(N)} = 0$;

3) $\Phi(\xi) = 0$;

4) $\Phi^2 = -id + \eta \otimes \xi$;

5) $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), X, Y \in \aleph(N)$.

So, we have prove the following statement.

Proposition. *An almost contact metric structure is induced on arbitrary oriented hypersurface of an almost Hermitian manifold.*

This fact was established by S. Sasaki [15] in another way.

3. The Main Result

Now, we can obtain the Cartan structural equations of an arbitrary almost contact metric structure induced on a hypersurface N in an almost Hermitian manifold.

We fix a point $p \in N$ and construct two basses of $T_p^C(M)$, where $T_p^C(M)$ is a complexification of tangent space of M at the point p:

$$b_1 = (\varepsilon_a, \varepsilon_{\hat{b}}, \varepsilon_n, \varepsilon_{\hat{n}}); b_2 = (\varepsilon_a, \varepsilon_{\hat{b}}, e_n, e_{\hat{n}}).$$

As $e_{\hat{n}}$ we denote the normal vector of N,

$$e_n = J(e_{\hat{n}}) = \xi_p,\ a, b, c, d = 1, \ldots, n-1;\ \hat{a} = a + n;\ \hat{n} = 2n.$$

If $(e_1, \ldots, e_n, Je_1, \ldots, Je_n)$ is an orthonormal basis of $T_p^C(M)$ adapted to the complex structure J, then

$$\varepsilon_a = \frac{e_a - iJe_a}{\sqrt{2}};\ \varepsilon_{\hat{a}} = \frac{e_a + iJe_a}{\sqrt{2}};$$

$$\varepsilon_n = \frac{e_n - iJe_n}{\sqrt{2}};\ \varepsilon_{\hat{n}} = \frac{e_n + iJe_n}{\sqrt{2}}.$$

The transition matrix from b_2 to b_1 is the following:

$$(C) = \left(\begin{array}{c|cc} I_{2n-2} & \multicolumn{2}{c}{0} \\ \hline 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{array}\right),$$

where I_{2n-2} is the $(2n-2)$-order unit matrix. So, we can deduce the inverse matrix:

$$(C^{-1}) = \left(\begin{array}{c|cc} I_{2n-2} & \multicolumn{2}{c}{0} \\ \hline 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{array}\right).$$

As (ω^i) and (θ^j) we denote the basses dual to b_2 and b_1, respectively; $i, j = 1, \dots, 2n$. We have:

$$\omega^i = \widetilde{C}^i_j \theta^j,$$

where $(\tilde{C}^i_j) = C^{-1}$. That is why we get:

$$\omega^\gamma = \theta^\gamma;\ \omega^n = \frac{1}{\sqrt{2}}(\theta^n + i\theta^{\hat{n}});\ \omega_n = \frac{1}{\sqrt{2}}(\theta^n - i\theta^{\hat{n}}),$$

where $\omega_n = \omega^{\hat{n}}$, $\gamma, \kappa = 1, \dots, n-1, n+1, \dots, 2n-1$.

Let us deduce the formula for θ^j:

$$\theta^\gamma = \omega^\gamma;\ \theta^n = \frac{1}{\sqrt{2}}(\omega^n + \omega_n);\ \theta^{\hat{n}} = \frac{1}{\sqrt{2}}(\omega^n - \omega_n).$$

The frames of the type (p, b_1) as well as of the type (p, b_2), $p \in N$ form the space of G-structure $\Upsilon_m, (m = 1, 2)$ with the structural group $G = U(n-1) \times \{e\}$.

The structural equations of the Riemannian connection ∇ on the spaces Υ_1 and Υ_2 of the structure are the following, respectively:

$$d\omega^i = \omega^i_j \wedge \omega^j;\ d\theta^i = \theta^i_j \wedge \theta^j,$$

where $\{\omega^i_j\}$ and $\{\theta^i_j\}$ are the Riemannian connection forms on the corresponding space of the G-structure.

If $\theta^i = C^i_j \omega^j$, then:

$$d\omega^k = (\tilde{C}^r_i \theta^i_j C^j_r) \wedge \omega^k.$$

From this relationship we have:

$$(\omega^i_j - \tilde{C}^i_k \theta^k_r C^r_j) \wedge \omega^j = 0.$$

Using the Cartan Lemma, we obtain:

$$\omega^i_j = \tilde{C}^i_k \theta^k_r C^r_j + C^i_{jk} \omega^k,$$

where $C^i_{jk} = C^i_{kj}$.

Taking into account that we consider the Riemannian connection [12], we can use that $\nabla g = 0$ (here $g = \{g_{ij}\}$ is the metric tensor). Let us write down the condition $\nabla g = 0$ in the basses b_1 and b_2:

1) $d\tilde{g}_{ij} + \tilde{g}_{kj}\omega^k_i + \tilde{g}_{ik}\omega^k_j = 0$;

2) $dg_{ij} + g_{kj}\theta^k_i + g_{ik}\theta^k_j = 0$.

Here $\{\tilde{g}_{kj}\}$ and $\{g_{kj}\}$ are the components of g in the basses b_1 and b_2, respectively. Taking into account

$$\tilde{g}_{ij} = g_{rs} C^r_i C^s_j,$$

we have:

$$g_{rs} C^r_k C^s_j \omega^k_i + g_{rs} C^r_i C^s_k \omega^k_j = 0,$$

or

$$g_{ms} C^s_j \theta^m_p C^p_i + g_{rm} C^r_i \theta^m_p C^p_j + (\tilde{g}_{kj} C^k_{il} + \tilde{g}_{ik} C^k_{jl})\omega^l = 0.$$

But according to [3],

$$g_{ms}\theta^m_p = -g_{pm}\theta^m_s,$$

and that is why we can conclude:

$$(\tilde{g}_{kj} C^k_{il} + \tilde{g}_{ik} C^k_{jl})\omega^l = 0.$$

Taking into account the basis forms ω^l are linearly independent, we get:

$$\tilde{g}_{kj} C^k_{il} + \tilde{g}_{ik} C^k_{jl} = 0.$$

Using the permutation $i \to j \to l \to i$, we obtain:

$$\tilde{g}_{kj} C^k_{il} = 0.$$

As the metric tensor $\{\tilde{g}_{kj}\}$ is not degenerated, we have:

$$C^k_{il} = 0.$$

As a result we can conclude that

$$\omega^i_j = \tilde{C}^i_k \theta^k_r C^r_j,$$

or

$$\theta^i_j = C^i_k \omega^k_r \tilde{C}^r_j.$$

On the space of the G-structure Υ_2, the hypersurface $N^{2n-1} \subset M^{2n}$ is determined by the following Pfaffian system [16]:

$$\theta^{\hat{n}} = 0.$$

By the exterior differentiation of this relationship we get:

$$\theta^{\hat{n}}_{\bar{\alpha}} \wedge \theta^{\hat{\alpha}} = 0, \ \bar{\alpha}, \bar{\beta} = 1, \dots, 2n-1.$$

According to the Cartan Lemma [12], we obtain:

$$\theta^{\hat{n}}_{\bar{\alpha}} = \sigma_{\bar{\alpha}\bar{\beta}} \theta^{\bar{\beta}},$$

where $\sigma_{\bar{\alpha}\bar{\beta}} = \sigma_{\bar{\beta}\bar{\alpha}}$ (here $\left\{\sigma_{\bar{\alpha}\bar{\beta}}\right\}$ are the components of the second fundamental form of the immersion of N into M).

Then we can write down the following condition:

$$C^{\hat{n}}_i \omega^i_j \tilde{C}^j_{\bar{\alpha}} = \sigma_{\bar{\alpha}\bar{\beta}} \theta^{\bar{\beta}} = \sigma_{\bar{\alpha}\beta} \theta^{\beta} + \sigma_{\bar{\alpha}n} \theta^n,$$

i.e.

$$C^{\hat{n}}_i \omega^i_j \tilde{C}^j_{\bar{\alpha}} = \sigma_{\bar{\alpha}\beta} \theta^{\beta} + \sigma_{\bar{\alpha}n} \frac{1}{\sqrt{2}} \left(\omega^n + \omega_n\right).$$

As the equality $\theta^{\hat{n}} = 0$ is equivalent to the condition $\omega_n = \omega^n = \frac{\theta^n}{\sqrt{2}}$, using the denotation $\theta = \theta^n$, we get:

$$C^{\hat{n}}_i \omega^i_j \tilde{C}^j_{\bar{\alpha}} = \sigma_{\bar{\alpha}\beta} \theta^{\beta} + \sigma_{\bar{\alpha}n} \theta.$$

Replacing θ by ω, we can rewrite this condition as follows:

1. $C^{\hat{n}}_i \omega^i_j \tilde{C}^j_{\alpha} = \sigma_{\alpha\beta} \omega^{\beta} + \sigma_{\alpha n} \omega;$

2. $C^{\hat{n}}_i \omega^i_j \tilde{C}^j_{n} = \sigma_{n\beta} \omega^{\beta} + \sigma_{nn} \omega.$

So,

1) $-\frac{i}{\sqrt{2}} \omega^n_{\alpha} + \frac{i}{\sqrt{2}} \omega^{\hat{n}}_{\alpha} = \sigma_{\alpha\beta} \omega^{\beta} + \sigma_{\alpha n} \omega,$

or

$\omega^n_{\alpha} - \omega^{\hat{n}}_{\alpha} = i\sqrt{2} \sigma_{\alpha\beta} \omega^{\beta} + i\sqrt{2} \sigma_{\alpha n} \omega.$

2) $-\frac{i}{2} \omega^n_n + \frac{i}{2} \omega^{\hat{n}}_n - \frac{i}{2} \omega^n_{\hat{n}} + \frac{i}{2} \omega^{\hat{n}}_{\hat{n}} = \sigma_{n\beta} \omega^{\beta} + \sigma_{nn} \omega.$

Note that

$$\omega^n_n = -\omega^{\hat{n}}_{\hat{n}}; \ \omega^{\hat{n}}_n = \omega_{nn} = 0; \ \omega^n_{\hat{n}} = \omega^{nn} = 0,$$

and consequently,

$$\omega^n_n = i \sigma_{n\beta} \omega^{\beta} + i \sigma_{nn} \omega.$$

Let

$$\alpha = a,$$

then

$$\omega_a^n - \omega_{na} = i\sqrt{2}\sigma_{ab}\omega^b + i\sqrt{2}\sigma_a^b\omega_b + i\sqrt{2}\sigma_{an}\omega.$$

Taking into account [2] that for an arbitrary almost Hermitian structure the following conditions hold:

$$\omega_{na} = B_{na}^b\omega_b + B_{na}^n\omega_n + \tilde{B}_{nab}\omega^b + \tilde{B}_{nan}\omega^n,$$

or

$$\omega_{na} = \tilde{B}_{nab}\omega^b + B_{na}^b\omega_b + (B_{na}^n + \tilde{B}_{nan})\omega^n,$$

and considering

$$\omega^n = \frac{1}{\sqrt{2}}\omega,$$

we have:

$$\omega_{na} = \left(\tilde{B}_{nab} + i\sqrt{2}\sigma_{ab}\right)\omega^b + \left(B_{na}^b + i\sqrt{2}\sigma_a^b\right)\omega_b + \left(\frac{1}{\sqrt{2}}B_{na}^n + \frac{1}{\sqrt{2}}\tilde{B}_{nan} + i\sqrt{2}\sigma_{an}\right)\omega.$$

Let

$$\alpha = \hat{a}.$$

Then

$$\omega_{\hat{a}}^n - \omega_{\hat{a}}^{\hat{n}} = i\sqrt{2}\sigma_{\hat{a}b}\omega^b + i\sqrt{2}\sigma_{\hat{a}}^b\omega_b + i\sqrt{2}\sigma_{\hat{a}n}\omega.$$

On the other hand,

$$\omega^{na} = \tilde{B}^{nab}\omega_b + B_b^{na}\omega^b + (B_n^{na} + \tilde{B}^{nan})\omega^n.$$

As a result we get:

$$\omega_n^a = \left(-\tilde{B}^{nab} + i\sqrt{2}\sigma^{an}\right)\omega_b + \left(-B_b^{na} + i\sqrt{2}\sigma_b^a\right)\omega^b + \left(-\frac{1}{\sqrt{2}}B_n^{na} - \frac{1}{\sqrt{2}}\tilde{B}^{nan} + i\sqrt{2}\sigma_n^a\right)\omega.$$

Taking into account this formulae, we can rewrite the structural equations of the almost contact metric structure induced on a hypersurface in an almost Hermitian manifold as follows:

$$d\omega^a = \omega_b^a \wedge \omega^b + \omega_n^a \wedge \omega^n + B_c^{ab}\omega^c \wedge \omega_b + B_c^{an}\omega^c \wedge \omega^n + B_n^{ac}\omega^n \wedge \omega_c + B^{abc}\omega_b \wedge \omega_c + B^{anc}\omega^n \wedge \omega_c + B^{acn}\omega_c \wedge \omega^n.$$

Substituting the value of ω_n^a and knowing that

$$\omega^n = \frac{1}{\sqrt{2}}\omega,$$

we have:

$$d\omega^a = \omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c$$
$$+ \left(\sqrt{2}B^{an}_b + i\sigma^a_b\right)\omega^b \wedge \omega + \left(-\sqrt{2}\tilde{B}^{nab} - \frac{1}{\sqrt{2}}B^{ab}_n - \frac{1}{\sqrt{2}}\tilde{B}^{abn} + i\sigma^{an}\right)\omega_b \wedge \omega.$$

Similarly,

$$d\omega_a = -\omega^b_a \wedge \omega_b + B^c_{ab}\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c$$
$$+ \left(\sqrt{2}B^b_{an} - i\sigma^b_a\right)\omega_b \wedge \omega + \left(-\sqrt{2}\tilde{B}_{nab} - \frac{1}{\sqrt{2}}\tilde{B}_{abn} - \frac{1}{\sqrt{2}}B^n_{ab} - i\sigma_{ab}\right)\omega^b \wedge \omega.$$

In conclusion, we obtain the third equation of Cartan structural equations. Taking into account the above mentioned relationship $\omega^n = \frac{1}{\sqrt{2}}\omega$, we get:

$$\frac{1}{\sqrt{2}}d\omega = d\omega^n = \omega^n_a \wedge \omega^a + \omega^n_n \wedge \omega^n + B^{na}_b \omega^b \wedge \omega_a$$
$$+ B^{na}_n \omega^n \wedge \omega_a + B^{nab}\omega_a \wedge \omega_b + B^{nnb}\omega^n \wedge \omega_b + B^{nbn}\omega_b \wedge \omega^n.$$

Substituting the formulae for ω^n_a, ω^n_n and ω^n, we can conclude:

$$d\omega = \sqrt{2}B_{nab}\omega^a \wedge \omega^b + \sqrt{2}B^{nab}\omega_a \wedge \omega_b + \left(\sqrt{2}B^{na}_b - \sqrt{2}B^a_{nb} - 2i\sigma^a_b\right)\omega^b \wedge \omega_a$$
$$+ \left(\tilde{B}_{nbn} + B^n_{nb} + i\sigma_{nb}\right)\omega \wedge \omega^b + \left(\tilde{B}^{nbn} + B^{nb}_n - i\sigma^b_n\right)\omega \wedge \omega_b.$$

So, we have proved the following theorem.

Theorem. *The Cartan structural equations of an almost contact metric structure induced on a hypersurface in an almost Hermitian manifold are the following:*

$$d\omega^a = \omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c$$
$$+ \left(\sqrt{2}B^{an}_b + i\sigma^a_b\right)\omega^b \wedge \omega + \left(-\sqrt{2}\tilde{B}^{nab} - \frac{1}{\sqrt{2}}B^{ab}_n - \frac{1}{\sqrt{2}}\tilde{B}^{abn} + i\sigma^{an}\right)\omega_b \wedge \omega;$$

$$d\omega_a = -\omega^b_a \wedge \omega_b + B^c_{ab}\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c$$
$$+ \left(\sqrt{2}B^b_{an} - i\sigma^b_a\right)\omega_b \wedge \omega + \left(-\sqrt{2}\tilde{B}_{nab} - \frac{1}{\sqrt{2}}\tilde{B}_{abn} - \frac{1}{\sqrt{2}}B^n_{ab} - i\sigma_{ab}\right)\omega^b \wedge \omega;$$

$$d\omega = \sqrt{2}B_{nab}\omega^a \wedge \omega^b + \sqrt{2}B^{nab}\omega_a \wedge \omega_b + \left(\sqrt{2}B^{na}_b - \sqrt{2}B^a_{nb} - 2i\sigma^a_b\right)\omega^b \wedge \omega_a$$
$$+ \left(\tilde{B}_{nbn} + B^n_{nb} + i\sigma_{nb}\right)\omega \wedge \omega^b + \left(\tilde{B}^{nbn} + B^{nb}_n - i\sigma^b_n\right)\omega \wedge \omega_b.$$

Remark that the obtained Cartan structural equations of an arbitrary almost contact metric structure on a hypersurface in an almost Hermitian manifold generalize the corresponding results for some almost contact metric structures of special type (see, for instance, [3, 4, 5, 17]).

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