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THE OPTIMALITY CONDITIONS OF A NONLINEAR PROGRAMMING MODEL FOR THE PROBLEM OF PACKING TRIANGLES

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Abstract: This paper considers a special two-dimensional strip packing problem in which the items to be packed are triangles. By the separation theory we formulate this problem as a nonlinear programming problem and establish the first-order optimality conditions for the NLP problem on the basis of quasidifferentiable theory.

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1. Introduction

The two-dimensional strip-packing problem (2SP) consists of allocating a set of n rectangles, without overlapping, to an open-ended bin of fixed width W and infinite height by minimizing the overall height of the bin. This paper concerns one of the extensions of the 2SP in which the items to be packed are triangles.

The background of this research is based on the extensive industrial applications. For example, in paper or cloth industries, irregular polygons have to be cut from a standardized unit (a roll of material), and the objective is to obtain the items by using the minimum roll length. Although the triangle packing is less complicated than the packing of irregular polygons, they are both

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NP-hard. Moreover, every irregular polygon can be partitioned or arbitrarily approximated by some triangles. Results of the triangle-packing problem will form a firm foundation for the further investigation of the more complicated irregular polygon problem.

In Section 2, we present a nonlinear programming model for this triangle-packing problem according to the separation theory. In Section 3, we establish the first-order optimality conditions for the NLP problem on the basis of quasidifferentiable theory.

2. A General Mathematical Model: Preliminary Knowledge

(I) By the separation theory, any two triangles Ω_1, Ω_2 in the plane are not overlapping if and only if there exists a line $a_1x_1 + a_2x_2 = b$ separating them, i.e.,

$$\max_{x\in\Omega_1}(a_1x_1 + a_2x_2) \le \min_{x\in\Omega_2}(a_1x_1 + a_2x_2).$$

It is easy to know the extreme value only can be obtained at the vertexes, for convenience, we use A_1^i, A_2^i, A_3^i to denote the vertexes of triangle Ω_i, $i = 1, 2$, let $a = (a_1, a_2)$, then triangles Ω_1, Ω_2 are not overlapping if and only if there exists a vector $a \in R^2$, satisfying

$$\max\{\langle a, A_1^1\rangle, \langle a, A_2^1\rangle, \langle a, A_3^1\rangle\} \le \min\{\langle a, A_1^2\rangle, \langle a, A_2^2\rangle, \langle a, A_3^2\rangle\},$$

it can be equally formulated as the following inequality :

$$\min_{\|a\|_\infty\le 1}\{\max\{\langle a, A_1^1\rangle, \langle a, A_2^1\rangle, \langle a, A_3^1\rangle\} - \min\{\langle a, A_1^2\rangle, \langle a, A_2^2\rangle, \langle a, A_3^2\rangle\}\} \le 0.$$

(II) Each triangle can be determined by its any two edges and the angle between them. Let A_1, A_2, A_3 denote the vertexes of triangle Ω, if $|A_1A_2| = l_1$, $|A_1A_3| = l_2$, $\angle A_2A_1A_3 = \theta$ and l_1, l_1, θ are known real numbers. In the coordinate system the position of Ω can be determined by the coordinate (x_1, x_2) of A_1 and the direction of $\overrightarrow{A_1A_2}$. Assuming the angle between $\overrightarrow{A_1A_2}$ and x axis is z, then the coordinate of vertex A_2 is $(x_1 + l_1\cos z, x_2 + l_1\sin z)$. If $\overrightarrow{A_1A_3}$ is generated by the anticlockwise (clockwise) rotating of $\overrightarrow{A_1A_2}$, then the coordinate of A_3 is $(x_1 + l_2\cos(z+\theta), x_2 + l_2\sin(z+\theta))$ $((x_1 + l_2\cos(z-\theta), x_2 + l_2\sin(z-\theta)))$.

Let us introduce some notations for the purpose of deriving the model of the packing problem:

L is the width of the bin, the set of n triangles is $\{\Omega_i : i \in J = \{1, \cdots, n\}\}$. Let A_1^i, A_2^i, A_3^i denote the vertexes of Ω_i, where $|A_1^iA_2^i| = l_1^i$, $|A_1^iA_3^i| = l_2^i$,

$\angle A_2^i A_1^i A_3^i = \theta^i$. We take the bottom left-hand corner of the bin as a position reference. Assuming the coordinate of A_1^i is (x_1^i, x_2^i), the angle between $\overrightarrow{A_1^i A_2^i}$ and x axis is z^i. For convenience we assume that $\overrightarrow{A_1^i A_3^i}$ $\overrightarrow{A_1^i A_2^i}$ is generated by the anticlockwise rotating of $\overrightarrow{A_1 A_2}$, thus the coordinates of A_2^i and A_3^i can be written as $(x_1^i + l_1^i \cos z^i, x_2^i + l_1^i \sin z^i)$, $(x_1^i + l_2^i \cos(z^i + \theta^i), x_2^i + l_2^i \sin(z^i + \theta^i))$ respectively. Now we can formulate the condition of the n triangles not overlapping as the following inequality:

$$\min_{\|a\|_\infty \le 1} \{\max\{\langle a, A_1^i\rangle, \langle a, A_2^i\rangle, \langle a, A_3^i\rangle\} - \min\{\langle a, A_1^j\rangle, \langle a, A_2^j\rangle, \langle a, A_3^j\rangle\}\} \le 0,$$

$$i, j \in J, i \ne j,$$

or

$$\min_{\|a\|_\infty \le 1} \max_{\substack{1\le k\le 3\\ 1\le h\le 3}} f_{(k,h)}^{i,j}(A^i, A^j, a) \le 0, \quad (i,j) \in H,$$

where $f_{(k,h)}^{i,j}(A^i, A^j, a) = \langle a, A_k^i - A_h^j\rangle, \quad H = \{(i,j) | i \in J, j \in J, i < j\}$.

We give the model of the problem of packing triangles:

$$\begin{aligned}
\min \ \max_{1\le i\le n} \max \quad & \{x_2^i,\ x_2^i + l_1^i \sin z^i,\ x_2^i + l_2^i \sin(z^i + \theta^i)\} \\
\text{subject to} \quad & \min_{\|a\|_\infty \le 1} \max_{\substack{1\le k\le 3\\ 1\le h\le 3}} f_{(k,h)}^{i,j}(A^i, A^j, a) \le 0, \quad (i,j) \in H, \\
& 0 \le x_1^i,\ x_1^i + l_1^i \cos z^i,\ x_1^i + l_2^i \cos(z^i + \theta^i) \le L, \quad i \in J, \\
& 0 \le x_2^i,\ x_2^i + l_1^i \sin z^i,\ x_2^i + l_2^i \sin(z^i + \theta^i), \quad i \in J.
\end{aligned}$$

To convert it into a general nonlinear programming problem, we introduce variable $w = (w^1, w^2, \cdots, w^n) \in ([0, L] \times [0, \infty] \times [0, 2\pi])^n$, where $w^i = (x_1^i, x_2^i, z^i)$, $i \in J$. Then the above model can be equally written as

NLP Model.

$$\begin{aligned}
& \min \ f(w) \\
& \text{subject to} f^{i,j}(w) \le 0, \quad (i,j) \in H, \\
& 0 \le g_1^i(w), g_2^i(w), g_3^i(w) \le L, \quad i \in J, \\
& 0 \le h_1^i(w), h_2^i(w), h_3^i(w), \quad i \in J.
\end{aligned} \tag{1}$$

where $f^{i,j}(w) = \min_{\|a\|_\infty \le 1} \max_{\substack{1\le k\le 3\\ 1\le h\le 3}} f_{(k,h)}^{i,j}(w^i, w^j, a)$, specifically,

$$f_{(1,1)}^{i,j}(w^i, w^j, a) = a_1(x_1^i - x_1^j) + a_2(x_2^i - x_2^j);$$
$$f_{(1,2)}^{i,j}(w^i, w^j, a) = a_1(x_1^i - x_1^j - l_1^j \cos z^j) + a_2(x_2^i - x_2^j - l_1^j \sin z^j);$$
$$f_{(1,3)}^{i,j}(w^i, w^j, a) = a_1(x_1^i - x_1^j - l_2^j \cos(z^j + \theta^j)) + a_2(x_2^i - x_2^j - l_2^j \sin(z^j + \theta^j));$$

$$
\begin{aligned}
f^{i,j}_{(2,1)}(w^i,w^j,a) &= a_1(x^i_1+l^i_1\cos z^i-x^j_1)+a_2(x^i_2+l^i_1\sin z^i-x^j_2);\\
f^{i,j}_{(2,2)}(w^i,w^j,a) &= a_1(x^i_1+l^i_1\cos z^i-x^j_1-l^j_1\cos z^j)+a_2(x^i_2+l^i_1\sin z^i-x^j_2\\
&\quad -l^j_1\sin z^j);\\
f^{i,j}_{(2,3)}(w^i,w^j,a) &= a_1(x^i_1+l^i_1\cos z^i-x^j_1-l^j_2\cos(z^j+\theta^j))+a_2(x^i_2+l^i_1\sin z^i-x^j_2\\
&\quad -l^j_2\sin(z^j+\theta^j));\\
f^{i,j}_{(3,1)}(w^i,w^j,a) &= a_1(x^i_1+l^i_2\cos(z^i+\theta^i)-x^j_1)+a_2(x^i_2+l^i_2\sin(z^i+\theta^i)-x^j_2);\\
f^{i,j}_{(3,2)}(w^i,w^j,a) &= a_1(x^i_1+l^i_2\cos(z^i+\theta^i)-x^j_1-l^j_1\cos z^j)+a_2(x^i_2+l^i_2\sin(z^i\\
&\quad +\theta^i)-x^j_2-l^j_1\sin z^j);\\
f^{i,j}_{(3,3)}(w^i,w^j,a) &= a_1(x^i_1+l^i_2\cos(z^i+\theta^i)-x^j_1-l^j_2\cos(z^j+\theta^j))+a_2(x^i_2\\
&\quad +l^i_2\sin(z^i+\theta^i)-x^j_2-l^j_2\sin(z^j+\theta^j));\\
g^i_1(w) &= x^i_1; \qquad\qquad g^i_2(w) = x^i_1+l^i_1\cos z^i;\\
g^i_3(w) &= x^i_1+l^i_2\cos(z^i+\theta^i); \qquad h^i_1(w) = x^i_2;\\
h^i_2(w) &= x^i_2+l^i_1\sin z^i; \qquad h^i_3(w) = x^i_2+l^i_2\sin(z^i+\theta^i).
\end{aligned}
$$

3. The First-Order Optimality Condition

It is necessary to analysis the qualities of the objective and constraint functions before giving the optimality condition of problem (1). It is easy to know that the constraint functions $g^i_k(w)$ and $h^i_k(w)$, $i=1,\cdots,n; k=\cdots 1,2,3$, are continuous, the derivatives of them are:

$$
\begin{aligned}
\nabla g^i_1(w) &= e_{3(i-1)+1};\\
\nabla g^i_2(w) &= e_{3(i-1)+1}-l^i_1\sin z^i e_{3i};\\
\nabla g^i_3(w) &= e_{3(i-1)+1}-l^i_2\sin(z^i+\theta^i)e_{3i};\\
\nabla h^i_1(w) &= e_{3(i-1)+2};\\
\nabla h^i_2(w) &= e_{3(i-1)+2}+l^i_1\cos z^i e_{3i};\\
\nabla h^i_3(w) &= e_{3(i-1)+2}+l^i_2\cos(z^i+\theta^i)e_{3i},
\end{aligned}
$$

where $e_k\in R^{3n}$ is the vector whose k-th element is 1, the others are zeros. The objective function

$$
\begin{aligned}
f(w) &= \max_{1\le i\le n}\max\{x^i_2,\, x^i_2+l^i_1\sin z^i,\, x^i_2+l^i_2\sin(z^i+\theta^i)\}\\
&= \max_{1\le i\le n}\max\{h^i_1(w),h^i_2(w),h^i_3(w)\}
\end{aligned}
$$

is piecewise smooth and subdifferentiable function, its subdifferential is

$$\underline{\partial} f(w) = \mathsf{co}\{\nabla h_k^i(w) | (i,k) \in I_1\},$$

where $I_1 = \{(i,k) \mid i = 1, \cdots, n;\ k = 1,2,3,\ f(w) = h_k^i(w)\}$.

The rather complicated are the constraint functions $f^{i,j}(w)$, $(i,j) \in H$. Let

$$f^{i,j}(w^i, w^j, a) = \max_{\substack{1\le k\le 3\\ 1\le h\le 3}} f_{(k,h)}^{i,j}(w^i, w^j, a),$$

then $f^{i,j}(w) = \min_{\|a\|_\infty \le 1} f^{i,j}(w^i, w^j, a)$, the directional derivative of $f^{i,j}(w)$ is

$$(f^{i,j})'(w; d) = \min_{a\in I_{ij}(w^i,w^j)} (f^{i,j})'(w^i, w^j, a; d),$$

where

$$I_{ij}(w^i, w^j) = \{a \in R^2 | \parallel a \parallel_\infty \le 1,\ f^{i,j}(w) = f^{i,j}(w^i, w^j, a)\}$$

$$(f^{i,j})'(w^i, w^j, a; d) = \max_{(k,h)\in R(w^i,w^j,a)} \nabla_{(w^i,w^j)} f_{(k,h)}^{i,j}(w^i, w^j, a)^\top d,$$

$$R(w^i, w^j, a) = \{(k,h) \mid f^{i,j}(w^i, w^j, a) = f_{(k,h)}^{i,j}(w^i, w^j, a), 1 \le k \le 3, 1 \le h \le 3\},$$

$$\nabla_{(w^i,w^j)} f_{(1,1)}^{i,j}(w^i, w^j, a) = a_1(e_{3(i-1)+1} - e_{3(j-1)+1}) + a_2(e_{3(i-1)+2} - e_{3(j-1)+2});$$

$$\nabla_{(w^i,w^j)} f_{(1,2)}^{i,j}(w^i, w^j, a) = a_1(e_{3(i-1)+1} - e_{3(j-1)+1} + l_1^j \sin z^j e_{3j}) + a_2(e_{3(i-1)+2} - e_{3(j-1)+2} - l_1^j \cos z^j e_{3j});$$

$$\nabla_{(w^i,w^j)} f_{(1,3)}^{i,j}(w^i, w^j, a) = a_1(e_{3(i-1)+1} - e_{3(j-1)+1} + l_2^j \sin(z^j + \theta^j) e_{3j}) + a_2(e_{3(i-1)+2} - e_{3(j-1)+2} - l_2^j \cos(z^j + \theta^j) e_{3j});$$

$$\nabla_{(w^i,w^j)} f_{(2,1)}^{i,j}(w^i, w^j, a) = a_1(e_{3(i-1)+1} - e_{3(j-1)+1} - l_1^i \sin z^i e_{3i}) + a_2(e_{3(i-1)+2} - e_{3(j-1)+2} + l_1^i \cos z^i e_{3i});$$

$$\nabla_{(w^i,w^j)} f_{(2,2)}^{i,j}(w^i, w^j, a) = a_1(e_{3(i-1)+1} - e_{3(j-1)+1} - l_1^i \sin z^i e_{3i} + l_1^j \sin z^j e_{3j}) + a_2(e_{3(i-1)+2} - e_{3(j-1)+2} + l_1^i \cos z^i e_{3i} - l_1^j \cos z^j e_{3j});$$

$$\nabla_{(w^i,w^j)} f_{(2,3)}^{i,j}(w^i, w^j, a) = a_1(e_{3(i-1)+1} - e_{3(j-1)+1} - l_1^i \sin z^i e_{3i} + l_2^j \sin(z^j + \theta^j) e_{3j}) + a_2(e_{3(i-1)+2} - e_{3(j-1)+2} + l_1^i \cos z^i e_{3i} - l_2^j \cos(z^j + \theta^j) e_{3j});$$

$$\nabla_{(w^i,w^j)} f_{(3,1)}^{i,j}(w^i, w^j, a) = a_1(e_{3(i-1)+1} - e_{3(j-1)+1} - l_2^i \sin(z^i + \theta^i) e_{3i}) + a_2(e_{3(i-1)+2} - e_{3(j-1)+2} + l_2^i \cos(z^i + \theta^i) e_{3i});$$

$$\nabla_{(w^i,w^j)} f_{(3,2)}^{i,j}(w^i, w^j, a) = a_1(e_{3(i-1)+1} - e_{3(j-1)+1} - l_2^i \sin(z^i + \theta^i) e_{3i} + l_1^j \sin z^j e_{3j})$$

$+ a_2(e_{3(i-1)+2} - e_{3(j-1)+2} + l_2^i \cos(z^i + \theta^i)e_{3i} - l_1^j \cos z^j e_{3j});$

$\nabla_{(w^i,w^j)} f^{i,j}_{(3,3)}(w^i, w^j, a) = a_1(e_{3(i-1)+1} - e_{3(j-1)+1} - l_2^i \sin(z^i + \theta^i)e_{3i} + l_2^j \sin(z^j$
$+ \theta^j)e_{3j}) + a_2(e_{3(i-1)+2} - e_{3(j-1)+2} + l_2^i \cos(z^i + \theta^i)e_{3i}$
$- l_2^j \cos(z^j + \theta^j)e_{3j}).$

We discuss the packing problem under the following assumption:

Assumption (a). $I_{ij}(w^i, w^j) = \{a \in R^2 | \parallel a \parallel_\infty \leq 1,\ f^{i,j}(w) = f^{i,j}(w^i, w^j, a)\}$ is a finite set.

Without loss of generality, suppose $I_{ij}(w^i, w^j) = \{a^1, a^2, \cdots, a^m\}$, now $f^{i,j}(w)$, $(i,j) \in H$ is quasidifferentiable function, its quasidifferential is $\mathcal{D}f^{i,j}(w) = [\underline{\partial} f^{i,j}(w), \overline{\partial} f^{i,j}(w)]$, where

$$\begin{aligned}
\underline{\partial} f^{i,j}(w) &= \sum_{\nu=1}^{m} \underline{\partial} f^{i,j}(w^i, w^j, a^\nu),\\
\overline{\partial} f^{i,j}(w) &= \mathsf{co}\{\overline{\partial} f^{i,j}(w^i, w^j, a^\nu) - \sum_{k\neq\nu} \underline{\partial} f^{i,j}(w^i, w^j, a^k) | k, \nu \in I_{ij}(w^i, w^j)\}.
\end{aligned}$$

According to $f^{i,j}(w^i, w^j, a) = \max\limits_{\substack{1\leq k\leq 3\\ 1\leq h\leq 3}} f^{i,j}_{(k,h)}(w^i, w^j, a)$ we know that $f^{i,j}(w^i, w^j, a)$ is piecewise smooth and subdifferentiable function,

$$\underline{\partial} f^{i,j}(w^i, w^j, a^\nu) = \mathsf{co}\{\nabla f^{i,j}_{(k,h)}(w^i, w^j, a^\nu) | (k,h) \in R(w^i, w^j, a^\nu)\},$$

$\overline{\partial} f^{i,j}(w^i, w^j, a^\nu) = 0$, so

$$\overline{\partial} f^{i,j}(w) = \mathsf{co}\{-\sum_{k\neq\nu} \underline{\partial} f^{i,j}(w^i, w^j, a^k) | k, \nu \in I_{ij}(w^i, w^j)\}.$$

Theorem 1. (First-Order Optimality Conditions for Model NLP) *If $\bar{w} \in ([0, L] \times [0, \infty] \times [0, 2\pi])^n$ is a local optimal solution of the NLP (1), then for any $v_{i,j} \in \overline{\partial} f^{i,j}(\bar{w}), (i,j) \in H$, there exist nonnegative real numbers which are not all zeros $\lambda_0(v)$, $\lambda_{i,j}(v)$, $(i,j) \in H, \alpha^i_k(v)$, $\beta^i_k(v)$, $\gamma^i_k(v)$, $i = 1, \cdots, n$; $k = 1, 2, 3$, there exist coefficient of convex combination η^i_k, $(i,k) \in I_1$ and $\xi^{i,j}_{(k,h),\nu}$, $(k,h) \in R(\bar{w}^i, \bar{w}^j, a^\nu)$, i.e. $\eta^i_k \geq 0$, $\xi^{i,j}_{(k,h),\nu} \geq 0$, $\sum\limits_{(i,k)\in I_1} \eta^i_k = 1$, $\sum\limits_{(k,h)\in R(\bar{w}^i,\bar{w}^j,a^\nu)} \xi^{i,j}_{(k,h),\nu} = 1$, such that*

$$\begin{aligned}
&\lambda_0(v) \sum_{(i,k)\in I_1} \eta^i_k \nabla h^i_k(\bar{w}) + \sum_{i=1}^{n} \sum_{k=1}^{3} [(\beta^i_k(v) - \alpha^i_k(v))\nabla g^i_k(\bar{w}) - \gamma^i_k(v) \nabla h^i_k(\bar{w})]\\
&+ \sum_{(i,j)\in H} \lambda_{i,j}(v)(\sum_{\nu=1}^{m} \sum_{(k,h)\in R(\bar{w}^i,\bar{w}^j,a^\nu)} \xi^{i,j}_{(k,h),\nu} \nabla f^{(i,j)}_{(k,h)}(\bar{w}^i, \bar{w}^j, a^\nu) + v_{i,j}) = 0,
\end{aligned}$$

$$\begin{array}{rcl} \lambda_{i,j}(v)f^{i,j}(\bar{w}) & = & 0, \\ \alpha_k^i(v)g_k^i(\bar{w}) & = & 0, \\ \beta_k^i(v)(g_k^i(\bar{w})-L) & = & 0, \\ \gamma_k^i(v)h_k^i(\bar{w}) & = & 0. \end{array}$$

Proof. According to Proposition 1.1 in reference [3], we know that for any $v_{i,j} \in \overline{\partial} f^{i,j}(\bar{w})$, $(i,j) \in H$, there exist nonnegative real numbers which are not all zeros $\lambda_0(v)$, $\lambda_{i,j}(v)$, $(i,j) \in H$, $\alpha_k^i(v)$, $\beta_k^i(v)$, $\gamma_k^i(v)$, $i = 1, \cdots, n$; $k = 1, 2, 3$, such that

$$\begin{aligned} 0 \in \lambda_0(v)\underline{\partial} f(\bar{w}) &+ \sum_{(i,j)\in H} \lambda_{i,j}(v)(\underline{\partial} f^{(i,j)}(\bar{w}) + v_{i,j}) \\ &+ \sum_{i=1}^{n}\sum_{k=1}^{3}[(\beta_k^i(v) - \alpha_k^i(v))\nabla g_k^i(\bar{w}) - \gamma_k^i(v)\nabla h_k^i(\bar{w})], \end{aligned}$$

$$\begin{array}{rcl} \lambda_{i,j}(v)f^{i,j}(\bar{w}) & = & 0, \\ \alpha_k^i(v)g_k^i(\bar{w}) & = & 0, \\ \beta_k^i(v)(g_k^i(\bar{w})-L) & = & 0, \\ \gamma_k^i(v)h_k^i(\bar{w}) & = & 0, \end{array}$$

i.e. there exist $\tau_0 \in \underline{\partial} f(\bar{w})$, $\tau_{i,j} \in \underline{\partial} f^{(i,j)}(\bar{w})$, $(i,j) \in H$, satisfying

$$\begin{aligned} \lambda_0(v)\tau_0 &+ \sum_{(i,j)\in H} \lambda_{i,j}(v)(\tau_{i,j} + v_{i,j}) \\ &+ \sum_{i=1}^{n}\sum_{k=1}^{3}[(\beta_k^i(v) - \alpha_k^i(v))\nabla g_k^i(\bar{w}) - \gamma_k^i(v)\nabla h_k^i(\bar{w})] = 0. \end{aligned}$$

Since

$$\tau_0 \in \underline{\partial} f(w) = \mathsf{co}\{\nabla h_k^i(w) | (i,k) \in I_1\},$$

$$\tau_{i,j} \in \underline{\partial} f^{i,j}(w) = \sum_{\nu=1}^{m} \underline{\partial} f^{i,j}(w^i, w^j, a^\nu),$$

where $\underline{\partial} f^{i,j}(w^i, w^j, a^\nu) = \mathsf{co}\{\nabla f^{i,j}_{(k,h)}(w^i, w^j, a^\nu) | (k,h) \in R(w^i, w^j, a^\nu)\}$, we have that there exist coefficient of convex combination η_k^i, $(i,k) \in I_1$ and $\xi^{i,j}_{(k,h),\nu}$, $(k,h) \in R(\bar{w}^i, \bar{w}^j, a^\nu)$, such that

$$\tau_0 = \sum_{(i,k)\in I_1} \eta_k^i \nabla h_k^i(\bar{w}),\ \tau_{i,j} = \sum_{\nu=1}^{m} \sum_{(k,h)\in R(\bar{w}^i,\bar{w}^j,a^\nu)} \xi^{i,j}_{(k,h),\nu} \nabla f^{(i,j)}_{(k,h)}(\bar{w}^i, \bar{w}^j, a^\nu),$$

by now we have proved the theorem. □

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