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ACCURACY OF THIRD-ORDER FINITE DIFFERENCE SCHEMES FOR SECOND DERIVATIVE APPROXIMATION ON NON-UNIFORM GRIDS

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Abstract: The Fourier error analysis of classical and compact differencing errors with third order accuracy for second derivative approximation on non-uniform grid system are presented. The results show that for uniform mesh compact difference scheme is superior to classical difference scheme, but for non-uniform the grid quality has stronger effects on compact scheme than on the classical scheme.

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1. Introduction

There are two finite difference schemes to be used in the discretization of the spatial derivatives in transitional and turbulent flow simulations. One is classical finite difference scheme (FDS) based on the expansion of Fourier serial. The other is the compact finite difference scheme (CFDS). A family of symmetric compact finite difference schemes with spectral-like resolution is developed by Lele [4]. Compact finite difference methods feature high-order accuracy with smaller stencils on uniform mesh. Hence, they are widely used for computa-

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tional fluid dynamics and computational acoustics, see [2-5]. However, compact schemes need to solve a matrix to obtain finite difference, it is a time consuming process. Classical difference schemes can be calculated directly. In order to save computational time, one prefers the finite difference calculated directly, in this case, one should know the accuracy of classical and compact difference methods.

For a study of near wall turbulence, in some time, the finite difference for second order derivative approximation is necessary. In turbulent flow simulations many flow fields are inhomogeneous in one or more direction. Because of the nature of turbulent flows and computing resource limitations, the use of non-uniform grid simulations is inevitable [2].

Most finite difference methods are written for uniform grid system. The extension of finite difference to non-uniform grid systems is not straightforward. In this study we assess the accuracy of third order finite difference schemes on non-uniform grids. The following two higher-order finite difference schemes are considered: a third order classical scheme (FDS), and the third order compact scheme (CFDS). Scheme accuracies are compared by using Fourier error analysis.

2. Third-Order Finite Difference Schemes

2.1. Third Order Classical Difference Schemes

For non-uniform mesh, the third order classical difference scheme for the second derivative may be rewritten in a general way:

$$f_i'' = A_i f_{i+1} + B_i f_{i+2} + C_i f_{i-1} + D_i f_{i-2} + E_i f_i \,. \tag{1}$$

Here the coefficients A_i, B_i, C_i, D_i, E_i are functions of the non-uniform mesh spacing $h_i = h_i - h_{i-1}$. We can obtain the value of former coefficients by matching the Taylor series of various orders, see [5]:

$$A_i = \frac{2(h_{i+1}h_i + h_{i+1}h_{i-1} - h_i^2 + 2h_i h_{i+2} - h_i h_{i-1} + h_{i+2}h_{i+1})}{(h_{i+1} + h_i + h_{i-1})(h_i + h_{i+1})h_{i+2}h_{i+1}} \,,$$

$$B_i = - \frac{2(2h_{i+1}h_i + h_{i+1}h_{i-1} - h_i^2 - h_i h_{i-1})}{(h_{i+1} + h_i + h_{i-1} + h_{i-2})(h_{i+1}^2 + 2h_{i+1}h_{i+2} + h_{i+2}^2 + h_{i+1}h_i + h_i h_{i+2})h_{i+2}} \,,$$

$$C_i = -\frac{2(h_{i+1}^2 - 2h_{i+1}h_i + h_{i+1}h_{i+2} - 2h_{i+1}h_{i-1} - h_ih_{i+2} - h_{i+2}h_{i-1}}{h_{i-1}(h_i + h_{i+2} + h_i)h_i(h_i + h_{i+1}}\,,$$

$$\begin{aligned}D_i = {} & (2(h_{i+1}h_{i+2} - 2h_{i+1}h_i - h_ih_{i+2} - h_{i+1}^2))/(h_{i-1}(2h_{i+1}h_{i-1}^2 \\ & + h_{i+2}h_{i-1}^2 + 3h_i^2h_{i-1} + 3h_ih_{i-1}^2 + h_i^3 + h_{i+1}^2h_{i-1} + h_{i-1}^3 + h_{i+1}^2h_i \\ & + 2h_{i+1}h_i^2 + 4h_{i+1}h_ih_{i-1} + 2h_{i+2}h_ih_{i-1} + h_i^2h_{i+2} + h_{i+2}h_{i+1}h_{i-1} + h_ih_{i+1}h_{i+2})\,,\end{aligned}$$

$$\begin{aligned}E_i = {} & -(2(2h_{i+1}h_{i-1} + h_{i+2}h_{i-1} - h_ih_{i-1} - h_{i+1}h_{i+2} + 2h_ih_{i+2} - h_{i+1}^2 \\ & - h_i^2 + 4h_{i+1}h_i))/(h_i + h_{i+1})h_i(h_{i+1} + h_{i+2})h_{i+1})\,. \qquad (2)\end{aligned}$$

2.2. Third Order Compact Difference Schemes

For non-uniform mesh, the compact approximation to the second derivative may be rewritten under the form:

$$a_if''_{i-1} + f''_i + \beta_if''_{i+1} = a_if_{i+1} + b_if_i + c_if_{i-1}\,, \qquad (3)$$

where

$$a_i = \frac{12h_i}{h_i^3 + h_{i+1}^3 + 4h_{i+1}h_i^2 + 4h_ih_{i+1}^2}, \quad b_i = -\frac{12}{h_{i+1}^2 + 3h_{i+1}h_i + h_i^2}\,,$$

$$c_i = \frac{12}{(h_i^3 + h_{i+1}^3 + 4h_{i+1}h_i^2 + 4h_ih_{i+1}^2)h_{i+1}},$$

$$\alpha_i = -\frac{h_{i+1}(-h_{i+1}h_i - h_i^2 + h_{i+1}^2)}{h_i^3 + h_{i+1}^3 + 4h_{i+1}h_i^2 + 4h_ih_{i+1}^2}\,,$$

$$\beta_i = \frac{(h_{i+1}^2 + h_{i+1}h_i - h_i^2)h_i}{h_i^3 + h_{i+1}^3 + 4h_{i+1}h_i^2 + 4h_ih_{i+1}^2}\,. \qquad (4)$$

3. Accuracy Analysis of the Differencing Error

In this section, we can obtain the accuracy of classical and compact difference schemes for second derivative approximation on non-uniform mesh. Fourier analysis, and the notion of the "modified wavenumber" provides a convenient means of quantifying the error associated with differencing schemes [6]. Consider the test function $f_j = e^{ikx_j}$ on a periodic domain. Discretize the function on a domain of length 2π using a non-uniform mesh. The mesh spacing is therefore given by $h_j = h_j - h_{j-1}$. The stretch ratio of the neighbor mesh increment is defined by $r = r_i = \frac{h_j}{h_{j-1}}$ and the exact values of the second derivative of f is

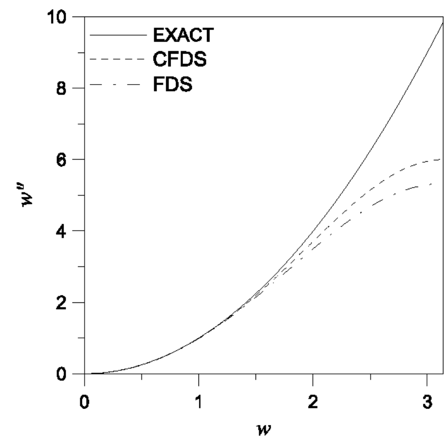

Figure 1: Comparision of error of the classical and compact difference methods on uniform grid system

$-k^2 e^{ikx_j}$. However, the numerically computed derivatives will be of $-k''^2 e^{ikx_j}$. The k'' is called the modified wavenumber for second derivative operator. The variable ω and ω'' related to r are functions of k and k'', respectively. The difference between ω and ω'' is a measure of error in the second derivative approximation. The classical and compact difference schemes are compared on five types of grid systems which have different mesh ratio and the results are shown in Figures 1-2.

From Figure 1, we can see that the third order compact difference scheme performs better than classical difference scheme with the same order accuracy on uniform grid system. In Figure 2, the investigation closely follows that of Lele [4]. The finite difference methods above are of central difference. Because of symmetry, the third order classical and compact methods contain no dissipation errors. However, the grid non-uniformity destroys this property, and the modified wave numbers become complex valued on non-uniform grids. As the grid non-uniformity increases, the ω'' is affected significantly. The deterioration of accuracy for the compact scheme is more serious than for the classical scheme. When the grid non-uniformity great than certain value, the advantage of using third order compact difference scheme over classical difference scheme is not clear. And the shortcoming of compact methods is that in such process a matrix must be solved to obtain finite difference, this is a time-consuming process, but classical methods can calculate directly. In this case, one should select classical difference scheme for second order derivative approximation.

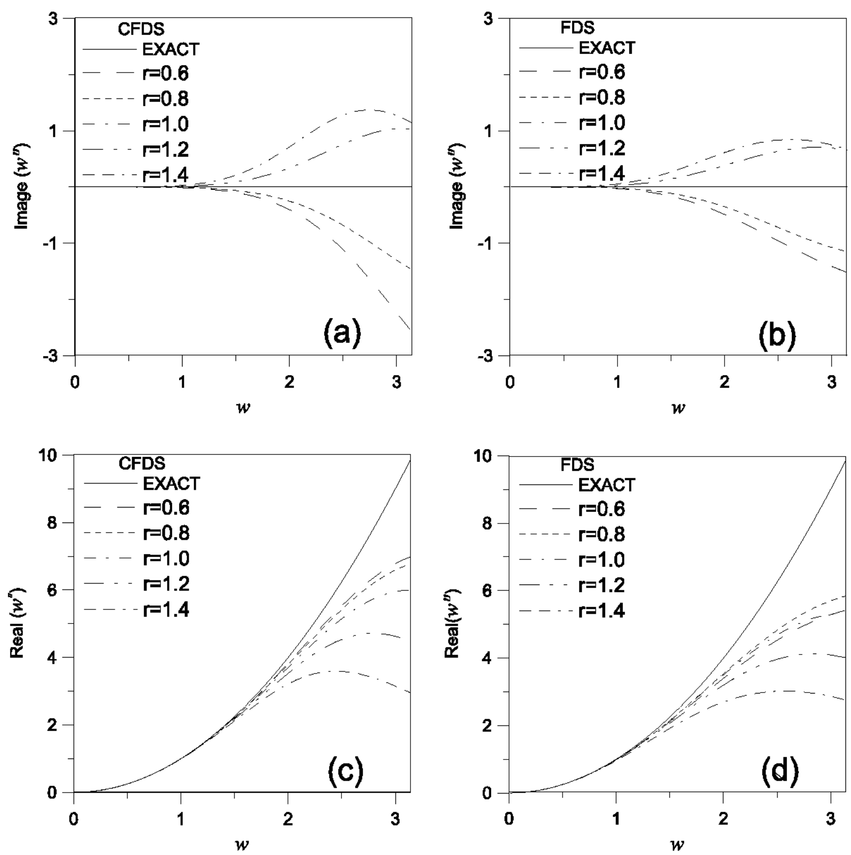

Figure 2: The modified wave numbers for second derivative approximations on four types of grid systems

4. Conclusions

In this paper, the formulations of a third order classical and compact difference scheme for the approximation of first derivatives on non-uniform meshes is presented. The accuracy of finite difference schemes on different grid systems is investigated by Fourier error analysis, we can get some conclusions that the third order compact difference scheme for second derivative approximation is superior to classical difference scheme for uniform grid system, but as the grid non-uniformity increases, the deterioration of accuracy for the compact method is more serious than for the classical method. The grid quality has stronger effects on the third order compact difference scheme than on the classical difference scheme with the same order accuracy. When the grid non-uniformity great than certain value, the advantage of using third order compact difference scheme over classical difference scheme is not clear. In this case, classical difference scheme is an advisable selection for second derivative approximation.

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