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SOME PROPERTIES OF MINIMAL SUBMANIFOLDS IN HYPERBOLIC SPACE

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Abstract: In this paper, we state a condition for minimal submanifolds in hyperbolic space to be totally geodesic submanifolds. Moreover, we conclude again this well known fact that compact submanifolds in hyperbolic space cannot be minimal submanifolds.

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1. Introduction

There are many results about minimal totally geodesic submanifolds in space forms. For example by some results of Di Scala (see [2], [3]) every minimal homogeneous submanifold of Euclidean space or hyperbolic space, is totally geodesic.

In this paper we study the totally geodesic property for minimal submanifolds, in two aspect. In one point of view, we introduce a theorem (Theorem 2) that contains a sufficient condition for a minimal submanifold in hyperbolic space, to be totally geodesic. On the other hand, we prove two distinct theorems (Theorem 1 and Theorem 3) that state a minimal submanifold M^n in hyperbolic space H^{n+1} with some additional conditions cannot be compact.

At first it needs to state some notions and notations. Let $(M^n, <,>)$ denoted an n-dimensional minimal compact submanifold in hyperbolic space H^{n+1} with induced Riemannian metric $<,>$ on M^n from H^{n+1}. Suppose that $\overline{\bigtriangledown}$ and $\bigtriangledown$ be Riemannian connections on M^n and H^{n+1}, respectively, and $\bigtriangledown^{\perp}$

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be the normal connection. Then we have,

$$\bigtriangledown_X Y = \overline{\bigtriangledown}_X Y + \Pi(X,Y), \quad \bigtriangledown_X N = -S_N X + \bigtriangledown^{\perp}_X Y, \tag{1.1}$$

where $X, Y \in \chi(M), N \in \Gamma(\nu)$ and Π is the second fundamental form. $\chi(M)$ is the Lie algebra of smooth vector fields on M, $\Gamma(\nu)$ is the space of smooth sections of the normal bundle ν of M and S_N is the shape operator of M corresponding to the normal vector field $N \in \Gamma(\nu)$ which satisfies,

$$< S_N X, Y > = < \Pi(X,Y), N > X, Y \in \chi(M), N \in \Gamma(\nu). \tag{1.2}$$

For a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ on M, since M is a minimal submanifold, we have,

$$\Sigma_i \Pi(e_i, e_i) = 0, \Sigma_i (\bigtriangledown \Pi)(X, e_i, e_i) = 0, \ \Sigma_i (\bigtriangledown^2 \Pi)(X, Y, e_i, e_i) = 0, \tag{1.3}$$

where tensors $\bigtriangledown \Pi$ and $\bigtriangledown^2 \Pi$ are first and second derivatives of second fundamental form of M^n, respectively.

Denote the Ricci tensor of M by Ric, then the Ricci operator O is a symmetric operator defined by

$$\mathrm{Ric}\,(X,Y) = < O(X), Y >, X, Y \in \chi(M).$$

The Gauss equation gives the following expression for the Ricci operator O of the minimal submanifold M,

$$O(X) = -(n-1)X - \Sigma_{i=1}^n S_{\Pi(e_i, X)} e_i, \tag{1.4}$$

where $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame on M. The scalar curvature $S = \Sigma_i \mathrm{Ric}\,(e_i, e_i)$ of the minimal submanifold of H^{n+1} is also given by,

$$S = -n(n-1) - ||\Pi||^2, \tag{1.5}$$

where $||\Pi||^2 = \Sigma_{i,j} ||\Pi(e_i, e_j)||^2$ is the squared norm of second fundamental form.

2. Some Basic Lemmas

In following lemmas, with assuming that there is a compact minimal submanifold in hyperbolic space, we reach to some results that have basic roles in proving Theorems 1 and 3.

Lemma 1. *Let M be an n-dimdensional minimal submanifold in the hyperbolic space H^{n+p}. Then,*

$$\int_M \{x_{n+p}^2[||\bigtriangledown\Pi||^2 + \Sigma_{ijk}[R^{\perp}(e_k, e_i, \Pi(e_j, e_k), \Pi(e_i, e_j)) - R(e_k, e_i, e_j, S_{\Pi(e_i,e_j)}e_k)] + \Sigma_{ij}\text{Ric}\,(e_i, S_{\Pi(e_i,e_j)}e_j)] + \frac{(2-n)}{2}x_{n+p}\Sigma_{ij}\bigtriangledown^{\perp}_{e_{n+p}}||\Pi(e_i, e_j)||^2\}dv = 0\,, \quad (2.1)$$

where R and $R^{\perp}$ are the curvature tensors corresponding to the connections $\bigtriangledown$ and $\bigtriangledown^{\perp}$, respectively.

Proof. Define $l : M \to R$, by $l = \frac{1}{2}||\Pi||^2$.If $\{e_1, e_2, \ldots, e_n\}$ be a local orthonormal frame and $\{e_{n+1}, e_{n+2}, \ldots, e_{n+p}\}$ be a set of local unit normal vector fields on M^n then the Laplacian of l is given by

$$\Delta l = (x_{n+p})^2\Sigma_{ijk} < (\bigtriangledown\Pi^2)(e_k, e_k, e_i, e_j), \Pi(e_i, e_j) > + \Sigma_{i,j,k}||\bigtriangledown\Pi(e_i, e_j, e_k)||^2 + (2-n)x_{n+p}\Sigma ij < (\bigtriangledown\Pi)(e_{n+p}, e_i, e_j, \Pi(e_i, e_j) > \,. \quad (2.2)$$

Since M is compact, l has compact support and we have $\int_M \Delta l dv = 0$.

On the other hand, we have the following equation,

$$< \bigtriangledown\Pi(e_{n+p}, e_i, e_j), \Pi(e_i, e_j) >= \frac{1}{2}\bigtriangledown^{\perp}_{e_{n+p}}||\Pi(e_i, e_j)||^2\,. \quad (2.3)$$

So using Ricci identity, Codazzi equation and (1.3) in (2.2) and integrating the resulting equation, we get the integral formula (2.1). □

Now, let $||R||^2$ be the squared curvature tensor field of M, which is

$$||R||^2 = \Sigma_{ijk}||R(e_i, e_j)e_k||^2,$$

where $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame on M. Therefore by Gauss equation and equation (1.3) we reach to following lemma immediately.

Lemma 2. *Let M be an n-dimdensional minimal submanifold in the hyperbolic space H^{n+p}. Then*

$$||R||^2 = -2n(n-1) + 2||S_{\Pi}||^2 - 2\Sigma_{ijk} < S_{\Pi(e_j,e_k)}e_i, S_{\Pi(e_i,e_k)}e_j > +4||\Pi||^2,$$

where $||S_{\Pi}||^2 = \Sigma_{ijk}||S_{\Pi(e_i,e_j)}e_k||^2$ and $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame on M.

Furthermore, we have following simplification for curvature tensor.

Lemma 3. *Let M be an n-dimdensional minimal submanifold in the hyperbolic space H^{n+p}. Then*

$$\Sigma_{ijk}R(e_k,e_i,e_j,S_{\Pi(e_i,e_j)}e_k)=\frac{1}{2}||R||^2-S-2||\Pi||^2\,, \tag{2.4}$$

where S is the scalar curvature of M and $\{e_1,e_2,\ldots,e_n\}$ is a local orthonormal frame on M.

Proof. By Gauss equation we have,

$$< S_{\Pi(e_k,e_j)}e_i, S_{\Pi(e_i,e_j)}e_k >=< S_{\Pi(e_i,e_j)}e_k, S_{\Pi(e_i,e_j)}e_k >$$
$$-< R_{(e_k,e_i)}e_j, S_{\Pi(e_i,e_j)}e_k > - < \delta_{ij}e_k, S_{\Pi(e_i,e_j)}e_k > + < \delta_{kj}e_i, S_{\Pi(e_i,e_j)}e_k > \ .$$

Now, using the above equation and equation (2.3) in (2.4) we have,

$$||R||^2=2S+4||\Pi||^2+2S_{\Pi(e_j,e_k)}e_iR((e_k,e_i,e_j,S_{\Pi(e_i,e_j)}e_k)\,.$$

So, we get the desired. □

3. Parallel Second Fundamental form

For proving the first theorem that state there is no compact minimal submanifold in hyperbolic space we use the notion of parallel second fundamental form.

Let M be an n-dimensional minimal submanifold in the hyperbolic space H^{n+p}, and $\{e_1,e_2,\ldots,e_n\}$ and $\{N_1,\ldots,N_p\}$ be a local orthonormal frame and a set of normals on M^n, respectively. We define the map $K^\perp : M\to R$ by

$$K^\perp=\Sigma_{ij\alpha\beta}[R^\perp(e_i,e_j,N_\alpha,N_\beta)]^2$$

and we call it the normal curvature of the minimal submanifold.

We also define a function $g : M\to R$ by,

$$g=2\Sigma_{\alpha<\beta}||S_\alpha||^2||S_\beta||^2,$$

where $S_\alpha=S_{N_\alpha}$ and $S_\beta=S_{N_\beta}$ are shape operators in directions vector fields N_α and N_β. We say M^n has parallel second fundamental form if the normal derivative of M^n in H^{n+1} be zero, i.e. $\bigtriangledown^\perp_X\Pi=0$ for all $X\in\chi(M)$.

Theorem 1. *Let M be an n-dimdensional minimal submanifold in the hyperbolic space H^{n+p}. If the normal curvature $K^\perp$ of M^n satisfies $K^\perp\geq g$ and M^n has parallel second fundamental form, then M^n cannot be compact.*

Proof. Let M^n be compact and $\{e_1, e_2, \dots, e_n\}$ be a local orthonormal frame on M. Then Gauss equation and equation (1.4) imply

$$S_{\Pi(e_j,e_k)}e_i = R(e_i,e_k)e_j - \delta_{ij}e_k - \delta_{kj}e_i + S_{\Pi(e_i,e_j)}e_k, \tag{3.1}$$

$$\Sigma_k S_{\Pi(e_j,e_k)}e_i = -(n-1)e_j - O(e_j). \tag{3.2}$$

Taking the inner product equation (3.1) with $S_{\Pi(e_i,e_j)}e_k$ we have

$$\begin{aligned}\Sigma_{ijk} &< S_{\Pi(e_i,e_j)}e_k, S_{\Pi(e_j,e_k)}e_k > \\ &= \Sigma_{ijk}R(e_i,e_k,e_j,S_{\Pi(e_i,e_j)}e_k) + ||\Pi||^2 + ||S_\Pi||^2 .\end{aligned} \tag{3.3}$$

In similar way equation (3.2) implies

$$\begin{aligned}\Sigma_{ijk} &< S_{\Pi(e_j,e_k)}e_k, S_{\Pi(e_i,e_j)}e_i > \\ &= -(n-1)||\Pi||^2 - \Sigma_{ij}\mathrm{Ric}\,(e_j, S_{\Pi(e_i,e_j)}e_i > .\end{aligned} \tag{3.4}$$

Using equations (3.3) and (3.4) in Ricci equation we get

$$\begin{aligned}&\Sigma_{ijk}R^{\perp}(e_k,e_i,\Pi(e_i,e_j),\Pi(e_j,e_k)) \\ &\quad = n||\Pi||^2 + ||S_\Pi||^2 + \Sigma_{ijk}R(e_i,e_k,S_{\Pi(e_i,e_j)}e_k) \\ &\qquad + \Sigma_{ij}\mathrm{Ric}\,(e_j, S_{\Pi(e_i,e_j)}e_i).\end{aligned} \tag{3.5}$$

For a local orthonormal frame $\{N_1, \dots, N_p\}$ of normals we have

$$\begin{aligned}\Sigma_{ij}\mathrm{Ric}\,(e_j, S_{\Pi(e_i,e_j)}e_i) &= \Sigma_{\alpha ij} < S_\alpha e_i, e_j > \mathrm{Ric}\,(e_j, S_\alpha e_i) \\ &= \Sigma_{\alpha i} < S_\alpha e_i, e_j > \mathrm{Ric}\,(\Sigma_j < S_\alpha e_i, e_j > e_j, , S_\alpha e_i) \\ &= \Sigma_{\alpha i}\mathrm{Ric}\,(S_\alpha e_i, S_\alpha e_i)\,.\end{aligned} \tag{3.6}$$

Using equation (1.4) in (3.6) we get

$$\Sigma_{ij}\mathrm{Ric}\,(e_j, S_{\Pi(e_i,e_j)}e_i) = -(n-1)||\Pi||^2 - \Sigma_{\alpha\beta}||S_\alpha S_\beta||^2\,. \tag{3.7}$$

Also we have

$$\Sigma_{ijk} < S_{\Pi(e_i,e_k)}e_i, S_{\Pi(e_i,e_k)}e_j = \Sigma_{ij\alpha\beta} < S_\alpha e_i, S_\beta e_j >< S_\alpha e_j, S_\beta e_i >, \tag{3.8}$$

and

$$\begin{aligned}\Sigma_{\alpha\beta}||S_\alpha S_\beta - S_\beta S_\alpha||^2 &= \Sigma_{ij\alpha\beta} < (S_\alpha S_\beta - S_\beta S_\alpha)(e_i), e_j) >^2 \\ &= -2\Sigma_{ij\alpha\beta}(< S_\alpha e_i, S_\beta e_j >< S_\alpha e_j, S_\beta e_i >) + 2\Sigma_{\alpha\beta}||S_\alpha S_\beta||^2.\end{aligned} \tag{3.9}$$

In above equation we use symmetry of S_α and S_β to reach $||S_\alpha S_\beta||^2 = ||S_\beta S_\alpha||^2$. Now, using equations (3.8) and (3.9) in (2.3) we get

$$||R||^2 = -2n(n-1) + 4||\Pi||^2 + 2||S_\Pi||^2 - 2\Sigma_{\alpha\beta}||S_\beta S_\alpha||^2 \,.$$

Furthermore, Lemma 3 implies,

$$2\Sigma_{ijk}R(e_k, e_i, e_j, S_{\Pi(e_i,e_j)}e_k) = -2n(n-1) + 2||S_\Pi||^2 - 2S + \Sigma_{\alpha\beta}||S_\alpha S_\beta - S_\beta S_\alpha||^2 - 2\Sigma_{\alpha\beta}||S_\beta S_\alpha||^2 \,. \quad (3.10)$$

The integral formula (2.1) in view of (3.5) takes the form

$$\int_M \{-x_{n+p}^2[||\bigtriangledown\Pi||^2 + ||S_\Pi||^2 + n||\Pi||^2 - 2\Sigma_{ijk}R(e_k, e_i, e_j, S_{\Pi(e_i,e_j)}e_k) + \Sigma_{ij}\mathrm{Ric}\,(e_i, S_{\Pi(e_i,e_j)}e_i)] + \frac{(2-n)}{2}x_{n+p}\Sigma_{ij}\bigtriangledown^\perp_{e_{n+p}}||\Pi(e_i,e_j)||^2\}dv = 0. \quad (3.11)$$

Because of

$$\bigtriangledown\Pi(e_k, e_i, e_j) = \bigtriangledown^\perp_{e_k}\Pi(e_i, e_j) - \Pi(\bigtriangledown_{e_k}e_i, e_j) - \Pi(e_i, \bigtriangledown^\perp_{e_k}e_j),$$

and our assumption, we have $\bigtriangledown\Pi(e_k, e_i, e_j) = 0$, for all $0 \le i, j, k \le n$. So $||\bigtriangledown\Pi||^2 = 0$ and (3.11) convert to

$$\int_M \{-x_{n+p}^2[||S_\Pi||^2 + n||\Pi||^2 - 2\Sigma_{ijk}R(e_k, e_i, e_j, S_{\Pi(e_i,e_j)}e_k) + \Sigma_{ij}\mathrm{Ric}\,(e_i, S_{\Pi(e_i,e_j)}e_i)] + \frac{(2-n)}{2}x_{n+p}\Sigma_{ij}\bigtriangledown^\perp_{e_{n+p}}||\Pi(e_i,e_j)||^2\}dv = 0. \quad (3.12)$$

Now, using relations (3.7), (3.10) and (1.5) in (3.12), we get:

$$\int_M \{-x_{n+p}^2[||S_\Pi||^2 + n||\Pi||^2 - \Sigma_{\alpha\beta}||S_\alpha S_\beta - S_\beta S_\alpha||^2]\}dv = 0. \quad (3.13)$$

On the other hand, in view of Ricci equation we have

$$K^\perp = \Sigma_{ij\alpha\beta}[R^\perp(e_i, e_j, N_\alpha, N_\beta)]^2 = \Sigma_{ij\alpha\beta} < [S_\alpha, S_\beta](e_i), e_j >^2 = \Sigma_{\alpha\beta}[\Sigma_{ij} < (S_\alpha S_\beta - S_\beta S_\alpha)(e_i), e_j >^2] = \Sigma_{\alpha\beta}||S_\alpha S_\beta - S_\beta S_\alpha||^2.$$

Also we have

$$||S_\Pi|| = \Sigma_{ijk}||S_{\Pi(e_i,e_j)}e_k||^2 = \Sigma_{ijk} < S_{\Pi(e_i,e_j)}e_k, S_{\Pi(e_i,e_j)}e_k >$$
$$= \Sigma_{ijk} < \Pi(e_i,e_j), \Pi(e_k, S_{\Pi(e_i,e_j)}e_k) >$$
$$= \Sigma_{ijk\alpha} < \Pi(e_i,e_j), N_\alpha >< N_\alpha, \Pi(e_k, S_{\Pi(e_i,e_j)}e_k) >$$
$$= \Sigma_{ij\alpha}||S_\alpha||^2 < S_\alpha e_i, e_j >= \Sigma_\alpha ||S_\alpha||^4 = (\Sigma_\alpha||S_\alpha||^2)^2 - 2\Sigma_{\alpha<\beta}||S_\alpha||^2||S_\beta||^2 ,$$

which implies

$$||S_\Pi||^2 = ||\Pi||^4 - 2\Sigma_{\alpha<\beta}||S_\alpha||^2||S_\beta||^2. \tag{3.15}$$

Using equations (3.14) and (3.15) in (3.13), we get

$$\int_M x_{n+p}^2\{-(n+||\Pi||^2)||\Pi||^2 + (g - K^\perp)\}dv = 0. \tag{3.16}$$

Now by using inequality $g \leq K^\perp$, since both terms in the left part of equation (3.16) are non-positive and sum of them have zero integral, both terms must be zero, i.e. $g = k^\perp$ and $(n+||\Pi||^2)||\Pi||^2 = 0$. Since $(n+||\Pi||^2) \neq 0$, it must be $||\Pi||^2 = 0$. Thus $\Pi = 0$, that states M is totally geodesic submanifold of H^{n+1} and we have a contradiction. □

In particular, if M^n be a hypersurface of H^{n+1}, for contradiction we do not need more assumption since we have

Corollary 1. *There is no n-dimensional compact minimal hypersurface M^n of H^{n+1} with parallel second fundamental form.*

Proof. Let M^n be compact. In the definition of $K^\perp$, since $\alpha = \beta = 1$ and we have

$$R^\perp(e_i, e_j, N_\alpha, N_\beta) =< R^\perp(e_i,e_j)N_\alpha, N_\alpha >= 0\,.$$

Therefore $K^\perp = 0$.

Now, if in $g = 2\Sigma_{\alpha<\beta}||S_\alpha||^2||S_\beta||^2$ take $\alpha = \beta = 1$, we get $g = 0$. Replacing above result in equation (3.10) we arrive at

$$\int_M -x_{n+p}^2(n+||\Pi||^2)||\Pi||^2 dv = 0.$$

This implies $||\Pi||^2 = 0$ or precisely $\Pi = 0$. So M^n is a totally geodesic hypersurface of H^{n+1}, i.e. contradiction. □

4. Flat Normal Curvature Tensor

A submanifold have flat normal curvature tensor if $R^{\perp} = 0$. In the first theorem of this section we study minimal submanifold in hyperbolic space whit flat normal curvature tensor, which is totally geodesic.

Theorem 2. *Let M^n be an n-dimensional minimal submanifold in hyperbolic space H^{n+p} with flat normal curvature tensor. If M^n have constant sectional curvature c, then $c \leq -1$ or M^n is a totally geodesic submanifold.*

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be a local orthonormal frame on M^n. Since M^n has constant sectional curvature c, by Gauss equation we have

$$(c+1)\{\delta_{jk}e_i - \delta_{ik}e_J\} = S_{\Pi(e_j,e_k)}e_i - S_{\Pi(e_i,e_k)}e_j\,, \tag{4.1}$$

where $1 \leq i, j.K \leq n$.

Taking the inner product with $S_{\Pi(e_j,e_k)}e_i$ and summing the resulting equation we get

$$\begin{aligned}&\Sigma_{ijk} < e_j, S_{\Pi(e_j,e_k)}e_i > \\ &\qquad = \Sigma_{ijk}\{< S_{\Pi(e_j,e_k)}e_i, S_{\Pi(e_j,e_k)}e_i > - < S_{\Pi(e_i,e_k)}e_j, S_{\Pi(e_j,e_k)}e_i >\}\,.\end{aligned}$$

The above equation simplifies to

$$(c+1)||\Pi||^2 = ||S_{\Pi}||^2 - \Sigma_{ijk} < S_{\Pi(e_i,e_k)}e_j, S_{\Pi(e_j,e_k)}e_i > \ . \tag{4.2}$$

On the other hand, since $R^{\perp} = 0$ and $R^{\perp}(X, Y, N_1, N_2) =< [S_{N_1}, S_{N_2}]$ $(X), Y >$, we have,

$$< [S_{\Pi(e_i,e_k)}, S_{\Pi(e_j,e_k)}](e_j), e_i >= 0\,.$$

Above equality, (3.2) and definition of symmetric operator O imply,

$$\begin{aligned}\Sigma_{ijk} < S_{\Pi(e_i,e_k)}e_j, S_{\Pi(e_j,e_k)}e_i >&= \Sigma_{ijk} < S_{\Pi(e_j,e_k)}e_j, S_{\Pi(e_i,e_k)}e_i > \\ &= \Sigma_{ik} < \Sigma_j S_{\Pi(e_j,e_k)}e_j, S_{\Pi(e_i,e_k)}e_i > \\ &= \Sigma_{ik} < -(n-1)e_k - O(e_k), S_{\Pi(e_i,e_k)}e_i > \\ &= -(n-1)||\Pi||^2 - \Sigma_{ik}\mathrm{Ric}\,(e_k, S_{\Pi(e_i,e_k)}e_i).\end{aligned} \tag{4.3}$$

Since the sectional curvature is equal constant c, therefore the Ricci curvature satisfies in the following equation

$$\mathrm{Ric}\,(e_k, S_{\Pi(e_i,e_k)}e_i) = (n-1)c < e_k, S_{\Pi(e_i,e_k)}e_i > \ .$$

Now we set the right side of above equation in (4.3). So we have

$$\Sigma_{ijk} < S_{\Pi(e_i,e_k)}e_j, S_{\Pi(e_j,e_k)}e_i >$$
$$= -(n-1)||\Pi||^2 - \Sigma_{ik}(n-1)c < e_k, S_{\Pi(e_i,e_k)}e_i) >$$
$$= -(n-1)(c+1)||\Pi||^2 \,. \tag{4.4}$$

Replacing (4.4) in (4.2) implies that

$$(c+1)||\Pi||^2 = ||S_\Pi||^2 + (c+1)(n-1)||\Pi||^2 \,,$$

or

$$||S_\Pi||^2 = -n(c+1)||\Pi||^2 \,. \tag{4.5}$$

Therefore (4.5) holds, if $c \leq -1$ or $c > -1$ and M^n be a totally geodesic submanifold of H^{n+p}. □

In the following theorem we state the second contradiction about a minimal submanifold in hyperbolic space that is also compact.

Theorem 3. *Let M be an n-dimensional non-negatively curved minimal submanifold in the hyperbolic space H^{n+p}. If M^n has flat normal connection, Then M^n cannot be compact.*

Proof. By the assumption, we have $R^\perp = 0$, therefore we have particular orthonormal frame $\{e_1, \ldots, e_n\}$ that for every $N \in \Gamma(\nu)$, S_N can be diagnolized. Let $\{N_1, \ldots, N_p\}$ be a local orthonormal frame of normal vector fields that for some smooth functions $\lambda_i^\alpha, \alpha = 1, \ldots, p$, we have $\lambda_\alpha(e_i) = \lambda_i^\alpha e_i$. Therefore

$$\Sigma_{ij}\mathrm{Ric}\,(e_i, S_{\Pi(e_i,e_j)}e_j) - \Sigma_{ijk}R(e_k, e_i, e_j, S_{\Pi(e_i,e_j)}e_k) = \Sigma_{\alpha jk}((\lambda_j^\alpha)^2 K_{kj}),$$

where $K_{kj} = R(e_k, e_j, e_j, e_k)$ is the sectional curvature of the plane section spanned by $\{e_k, e_j\}$. Therefore we have

$$\Sigma_{ij}\mathrm{Ric}\,(e_i, S_{\Pi(e_i,e_j)}e_j) - \Sigma_{ijk}R(e_k, e_i, e_j, S_{\Pi(e_i,e_j)}e_k)$$
$$= \frac{1}{2}[\Sigma_{\alpha jk}((\lambda_j^\alpha - \lambda_k^\alpha)^2 K_{kj})] \geq 0. \tag{4.6}$$

Now, if M^n be compact using (2.1), assumption $R^\perp = 0$ and (3.6), we get: $||\bigtriangledown\Pi|| = 0$ and

$$\Sigma_{ij}\mathrm{Ric}\,(e_i, S_{\Pi(e_i,e_j)}e_j) - \Sigma_{ijk}R(e_k, e_i, e_j, S_{\Pi(e_i,e_j)}e_k) = 0. \tag{4.7}$$

In equation (3.5), if $R^\perp = 0$, then by second equation of (4.7), we arrive at:

$$||S_\Pi||^2 + n||\Pi||^2 = 0. \tag{4.8}$$

Thus we have,

$$\Sigma_\alpha(||S_\alpha||^2 + \frac{n}{2})^2 = \frac{n^2 p}{4}. \tag{4.9}$$

Now Schwartz inequality yields

$$\Sigma_\alpha(||S_\alpha||^2 + \frac{n}{2})^2 \geq \frac{1}{p}[\Sigma_\alpha(||S_\alpha||^2 + \frac{n}{2})]^2 \ = \frac{1}{p}(||\Pi||^2 + \frac{np}{2})^2. \tag{4.10}$$

Therefore (4.9) and (4.10) imply

$$\frac{n^2 p}{4} \geq \frac{1}{p}(((||\Pi||^2 + \frac{np}{2})^2 .$$

Since two sides of above inequality are positive, we have:

$$\frac{np}{2} \geq ||\Pi||^2 + \frac{np}{2}.$$

So $||\Pi|| = 0$, that means we arrive at the contradiction, M^n is a totally geodesic submanifold. □

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