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THE NUMERICAL SOLUTION OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS USING NONPOLYNOMIAL SPLINE

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Abstract: In this paper, we develop a class of methods for the numerical solution to singularly perturbed boundary value problems (SPBVPs) using nonpolynomial spline. The present approach leads to a generalized scheme that has third-order and fourth-order convergence depending on the choice of the parameters α, β and ω's involved in the method. Our scheme leads to a tridiagonal system of linear equations. Convergence analysis of the methods is discussed. Two numerical examples are included to validate the practical usefulness and the superiority of our methods.

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1. Introduction

We consider a second order singularly perturbed boundary value problem:

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$$-\varepsilon y^{(2)} + f(x)y = g(x), \quad f(x) > 0, \quad x \in [a, b], \tag{1.1}$$

$$y(a) - A_1 = y(b) - A_2 = 0, \tag{1.2}$$

where A_i, $i = 1, 2$ are finite real constants, ε is a parameter such that $0 < \varepsilon \ll 1$ and $f(x)$ and $g(x)$ are smooth known functions.

The solution of singularly perturbed boundary value problems (SPBVPs) exhibits a multiscale character. That is, there is a thin layer, where the solution varies rapidly, while away from the layer the solution behaves regularly and varies slowly.

This class of (SPBVPs) has recently gained importance in the literature because they commonly occur in various fields of science and engineering such as fluid mechanics, fluid dynamics, elasticity, quantum mechanics, chemical reactor theory, convection diffusion processes and optimal control. Also, the numerical treatment of (SPBVPs) gives major computational difficulties and the standard methods do not yield accurate results for all x when ε is very small relative to the mesh size h that is used for the discretization of SPBVPs. A variety of numerical methods are available in the literature to solve SPBVPs for second order ordinary differential equations. For details, we refer to survey article by Kadalbajoo and Patidar [4]. The application of spline and nonpolynomial spline for the numerical solution of SPBVPs and regular boundary value problems respectively have been described in many papers [1], [9]-[12], [13], [17]-[18]. Niijima [7], [8] established convergent second order finite difference schemes. Surla et al [14], [16] generated a difference scheme via spline in tension and exponential spline and obtained the error estimate. Kadalbajoo et al [3] discussed a variable mesh difference scheme using spline. Rashidinia et al [13] and Tariq Aziz et al [17] have solved the problem (1.1), (1.2) using spline in compression. Chandra et al [1] demonstrated an optimal cubic B-spline collocation method solving the above problem by decomposing the domain into three non-overlapping subdomains. Kadalbajoo et al [5] considered cubic B-spline collocation to generate second order convergent method for solving SPBVPs by dividing the domain into three non-overlapping subdomains and selecting the step size h to generate more points in the boundary layers region than outside of it.

In the present paper, quadratic nonpolynomial spline approach that has a polynomial part and a trigonometric part is used to develop generalized scheme for the numerical solution of (1.1), (1.2). The comparative advantage of our scheme is that quadratic nonpolynomial splines need less coefficients evaluations as compared with Rashidinia et al [13], Tariq Aziz et al [17] and Kadalbajoo et al [3]. Also, the proposed scheme allows us to get different methods of different orders according to the choice of the parameters α, β and ω with the same computational effort. This method is an extension of a previous work of the authors [9]-[12], [18] when second order, fourth order, sixth order and system of second order two point boundary value problems are presented.

The paper is organized as follows: In Section 2, we derive our method. In Section 3, we formulate our method in matrix form. Convergence analysis for fourth order method is established in Section 4. Finally, numerical results and discussion are presented in Section 5.

2. Derivation of the Method

We introduce a finite set of grid points x_i by dividing the interval $[a, b]$ into n equal parts.

$$x_i = a + ih,\ x_0 = a,\ x_n = b \text{ and } h = \frac{b-a}{n},\ i = 0, 1, \ldots, n. \tag{2.1}$$

Let $y(x)$ be the exact solution of the system (1.1) and S_i be an approximation to $y_i = y(x_i)$ obtained by the spline function $Q_i(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}). Each nonpolynomial spline segment $Q_i(x)$ has the form:

$$Q_i(x) = a_i \cos k(x - x_i) + b_i \sin k(x - x_i) + c_i, \quad i = 0, 1, \ldots, n\,, \tag{2.2}$$

where a_i, b_i and c_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method. Thus, our quadratic nonpolynomial spline is now defined by the relations:

(i) $S(x) = Q_i(x)$, $x \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n$, and

(ii) $S(x) \in C^{\infty}[a, b]$. (2.3)

The three coefficients in (2.2) need to be obtained in terms of $S_{i+1/2}$, D_i and $F_{i+1/2}$, where:

(i) $Q_i(x_{i+1/2}) = S_{i+1/2}$,

(ii) $Q_i^{(1)}(x_i) = D_i$, (2.4)

(iii) $Q_i^{(2)}(x_{i+1/2}) = F_{i+1/2}$.

We obtain via a straightforward calculation

$$a_i = \frac{-\sec(\theta/2)}{k^2} F_{i+1/2} - \tfrac{\tan(\theta/2)}{k} D_i,\ b_i = \tfrac{1}{k} D_i,\ c_i = S_{i+1/2} - \tfrac{1}{k^2} F_{i+1/2}, \tag{2.5}$$

where $\theta = kh$ and i=0,1,. . . , n-1. Now from the continuity conditions (ii) in (2.3), that is the continuity of quadratic nonpolynomial spline $S(x)$ and its first derivative at the point (x_i, S_i), where the two quadratics $Q_{i-1}(x)$ and $Q_i(x)$ join. We have $Q^{(\lambda)}_{i-1}(x) = Q^{(\lambda)}_i(x)$, λ =0,1.

Using equations (2.2) and (2.5) yield the relations

$$\frac{1}{k}\tan(\theta/2)\left(D_i + D_{i-1}\right) = \left(S_{i+1/2} - S_{i-1/2}\right) + \frac{1}{k^2}[F_{i+1/2}(1-\sec(\theta/2)) + F_{i-1/2}(\cos(\theta)\sec(\theta/2)-1)], \quad (2.6)$$

$$(D_i - D_{i-1}) = \frac{2}{k}F_{i-1/2}\sin(\theta/2)\,. \quad (2.7)$$

From equations (2.6) and (2.7) we get the following relation

$$\left(S_{i+1/2} - 2S_{i-1/2} + S_{i-3/2}\right) = h^2\left(\alpha F_{i+1/2} + \beta F_{i-1/2} + \alpha F_{i-3/2}\right), \quad (2.8)$$

$i = 2, 3, \ldots, n-1$, where $\alpha = \frac{\sec(\theta/2)-1}{\theta^2}$ and $\beta = \frac{4\sec(\theta/2)\sin^2(\theta/2)+2(1-\sec(\theta/2)}{\theta^2}$.

The relation (2.8) gives $n-2$ linear algebraic equations in the n unknowns $S_{i+1/2}$, $i = 0, 1, 2, \ldots, n-1$, so we need two more equations, one at each end of the range of integration for the direct computation of $S_{i+1/2}$. These two equations are deduced by Taylor series and the method of undetermined coeffients. These equations are

$$2S_0 - 3S_{1/2} + S_{3/2} = h^2(\omega_0 F_0 + \omega_1 F_{1/2} + \omega_2 F_{3/2} + \omega_3 F_{5/2}), \quad (2.9)$$

Similarly, we get the second end condition as:

$$2S_n - 3S_{n-1/2} + S_{n-3/2} = h^2(\omega_0 F_n + \omega_1 F_{n-1/2} + \omega_2 F_{n-3/2} + \omega_3 F_{n-5/2}), \quad (2.10)$$

where ω_i's will be determined later to get the required order of accuracy.

To get the local truncation errors t_i, $i = 2, 3, \ldots, n-1$, associated with the scheme (2.8), first we rewrite the scheme (2.8) in the form:

$$\left(y_{i+1/2} - 2y_{i-1/2} + y_{i-3/2}\right) = h^2\left(\alpha y^{(2)}_{i+1/2} + \beta y^{(2)}_{i-1/2} + \alpha y^{(2)}_{i-3/2}\right) + t_i, \quad i = 2, 3, \ldots, n-1. \quad (2.11)$$

The terms $y_{i-1/2}, y^{(2)}_{i-1/2}$, etc., are expanded around the point x_i using Taylor series and the expressions for t_i, $i = 2, 3, \ldots, n-1$ can be obtained. Expressions for t_i, $i = 1, n$ are obtained in a similar manner. The local truncation errors t_i, $i = 2, 3, \ldots, n-1$ associated with the scheme (2.8) are:

$$t_i = h^2(1-2\alpha-\beta)y_i^{(2)} + h^3(\alpha+\frac{\beta}{2}-\frac{1}{2})y_i^{(3)} + h^4(\frac{5}{24}-\frac{5}{4}\alpha-\frac{\beta}{8})y_i^{(4)}$$
$$- h^5(\frac{1}{16}-\frac{26\alpha}{48}-\frac{\beta}{48})y_i^{(5)} + h^6(\frac{91}{5760}-\frac{82\alpha}{384}-\frac{\beta}{384})y_i^{(6)} + O(h^7),$$
$$i = 2,3,\dots,n-1. \quad (2.12)$$

The scheme (2.8)-(2.10) gives rise to a class of methods according to the choice of α, β and ω 's as follows:

Class of Methods. For $(\alpha,\beta) = (\frac{1}{12},\frac{5}{6})$.

1. Third Order Convergence Method. for $(\omega_0,\omega_1,\omega_2)/d_1 = (-2,33,5)$ where $d_1 = 48$. Then the local truncation errors are:

$$t_i = \begin{cases} \frac{-1}{96}h^5 y_i^{(5)} + O(h^6), i = 1, n, \\ \frac{-1}{240}h^6 y_i^{(6)} + O(h^7),\ i = 2,3,\dots,n-1. \end{cases} \quad (2.13)$$

2. Fourth Order Convergence Method. For $(\omega_0,\omega_1,\omega_2,\omega_3)/d_2 = (-10,750,175,-15)$, where $d_2 = 1200$. Then the local truncation errors are:

$$t_i = \begin{cases} \frac{19}{5120}h^6 y_i^{(6)} + O(h^7), i = 1, n, \\ \frac{-1}{240}h^6 y_i^{(6)} + O(h^7),\ i = 2,3,\dots,n-1. \end{cases} \quad (2.14)$$

Remark. 1. As $k \to 0$ then $(\alpha,\beta) = (\frac{1}{8},\frac{6}{8})$ and the scheme (2.8) is then reduced to second order method based on quadratic polynomial spline, see [9].

2. When $(\alpha,\beta) = (\frac{1}{24},\frac{22}{24})$, then the scheme (2.8) is then reduced to second order method based on cubic polynomial spline, see [9].

3. Nonpolynomial Spline Solutions

The spline solution of (1.1) with the boundary condition (1.2) is based on the linear equations given by (2.8)-(2.10).

Let $Y = (y_{i+1/2})$, $S = (S_{i+1/2})$, $T = (t_i)$, $E = (e_{i+1/2}) = Y - S$, $C = (C_i)$, be n dimensional column vectors. Then we can rewrite equations (2.8)-(2.10) in the standard matrix equations form:

(i) $NY = C + T$,
(ii) $NS = C$, (3.1)
(iii) $NE = T$.

We also have

$$N = \varepsilon N_0 + h^2 BF, \quad F = \text{diag}(f_i), \quad (3.2)$$

where N_0 and B are banded matrices. The symmetric matrix N_0 has the form:

$$\varepsilon N_0 = \begin{pmatrix} 3\varepsilon & -\varepsilon & & & & & \\ -\varepsilon & 2\varepsilon & -\varepsilon & & & & \\ & -\varepsilon & 2\varepsilon & -\varepsilon & & & \\ & & & \ddots & & & \\ & & & -\varepsilon & 2\varepsilon & -\varepsilon & \\ & & & & -\varepsilon & 2\varepsilon & -\varepsilon \\ & & & & & -\varepsilon & 3\varepsilon \end{pmatrix} \tag{3.3}$$

and the matrix B has the form

$$B = \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 & & & & \\ \alpha & \beta & \alpha & & & & \\ & \alpha & \beta & \alpha & & & \\ & & & \ddots & & & \\ & & & \alpha & \beta & \alpha & \\ & & & & \alpha & \beta & \alpha \\ & & & & \omega_3 & \omega_2 & \omega_1 \end{pmatrix}. \tag{3.4}$$

For the vector C, we have

$$C_i = \begin{cases} h^2[\omega_0(-A_1 f_0 + g_0) + \omega_1 g_{1/2} + \omega_2 g_{3/2} + \omega_3 g_{5/2}] - 2A_1, & i = 1\,, \\ h^2(\alpha\ g_{i+1/2} + \beta\ g_{i-1/2} + \alpha\ g_{i-3/2}), & i = 2, 3, \ldots, n-1\,, \\ h^2[\omega_0(-A_2 f_n + g_n) + \omega_1 g_{n-1/2} + \omega_2 g_{n-3/2} + \omega_3 g_{n-5/2}] - 2A_2, & i = n\,. \end{cases} \tag{3.5}$$

4. Convergence Analysis

Our main purpose now is to derive a bound on $\|E\|_\infty$ for SPBVP (1.1), (1.2). We now turn back to the error equation (iii) in (3.1) and rewrite it in the form:

$$E = N^{-1} T\,, \tag{4.1}$$

where $N = \varepsilon N_0 + h^2 BF$ and let $|f_i| \le R$, where $R > 0$. Let $n_{(i,j)}$ be the (i, j)-th element of the matrix N.

Thus the row sums $N_1, N_2, \ldots, N_n$ of the matrix N satisfies:

$$N_i = \sum_{j=1}^{n} n_{(i,j)} = 2\varepsilon + (\omega_1 + \omega_2 + \omega_3)h^2 R, \quad i = 1, n,$$

$$N_i = \sum_{j=1}^{n} n_{(i,j)} = (2\alpha + \beta)h^2 R, \quad i = 2, 3, \ldots, n-1. \tag{4.2}$$

Then the matrix N becomes irreducible and monotone and then it follows N^{-1} exists and $N^{-1} \geq 0$, see [2]. Thus from the error equation (4.1), we have

$$\|E\|_\infty \leq \left\|(\varepsilon N_0 + h^2 BF)^{-1}\right\|_\infty \|T\|_\infty . \tag{4.3}$$

Let $(\varepsilon N_0 + h^2 BF)^{-1}_{(i,j)}$ be the (i,j)-th element of the matrix $(\varepsilon N_0 + h^2 BF)^{-1} = N^{-1}$.

Define

$$\left\|(\varepsilon N_0 + h^2 BF)^{-1}_{(i,j)}\right\|_\infty = \max_{1\leq j\leq n} \sum_{i=1}^{n} \left|(\varepsilon N_0 + h^2 BF)^{-1}_{(i,j)}\right|. \tag{4.4}$$

Then from the theory of matrices

$$\sum_{j=1}^{n} (\varepsilon N_0 + h^2 BF)^{-1}_{(i,j)} . N_{(i,j)} = 1, \quad j = 1, 2, \ldots, n. \tag{4.5}$$

Hence

$$\sum_{j=1}^{n} (\varepsilon N_0 + h^2 BF)^{-1}_{(i,j)} \leq \frac{1}{\min\limits_{1\leq i\leq n} N_i} = \frac{1}{\eta h^2}, \tag{4.6}$$

where $\varepsilon = O(h^2)$ and $\eta = (2\alpha + \beta)R$ is constant independent of h.

From equation (2.14), we get $\|T\|_\infty = \frac{h^6}{240} M_6$, where $M_6 = \max\limits_{a\leq x\leq b} \left|y^{(6)}(x)\right|$. Then

$$\|E\|_\infty \leq \left\|(\varepsilon N_0 + h^2 BF)^{-1}\right\|_\infty \|T\|_\infty \leq \frac{1}{\eta h^2} \frac{h^6}{240} M_6 \,, \tag{4.7}$$

$$\|E\|_\infty \leq \frac{M_6}{240\eta} h^4 = G_4 h^4, \tag{4.8}$$

where the constant $G_4 = \frac{M_6}{240\eta}$.

However, from equation (3.5), $\|T\|_\infty = \frac{h^5}{96} M_5$, where $M_5 = \max\limits_{a\leq x\leq b} \left|y^{(5)}(x)\right|$. Then

$$\|E\|_\infty \leq \frac{M_5}{96\eta} h^3 = G_3 h^3, \tag{4.9}$$

where the constant $G_3 = \frac{M_5}{96\eta}$.

We summarize the above results in the next theorem.

Theorem 4.1. *Let $y(x)$ be the exact solution of the SPBVPs (1.1) with the boundary condition (1.2) and let y_i, $i = 1, 2, \ldots, n$, satisfy the discrete boundary value problem (ii) in (3.1). Further, if $e_{i+1/2} = y_{i+1/2} - S_{i+1/2}$, then*

$$\|E\|_\infty \cong O(h^4), \qquad \|E\|_\infty \cong O(h^3).$$

Thus our method is of a fourth order and third order convergent methods given by (4.8) and (4.9) respectively.

5. Numerical Examples and Concluding Remarks

We now consider two numerical examples to illustrate the comparative performance of our methods (ii) in (3.1) over other existing methods. All calculations are implemented by *Matlab 6.5*.

Example 1. Consider the SPBVPs, see [13],

$$\varepsilon y^{(2)} + y = -\cos^2(\pi x) - 2\varepsilon\pi^2 \cos(2\pi x), y(0) = \ y(1) = 0. \tag{5.1}$$

The analytical solution of (5.1) is

$$y(x) = \frac{e^{-(1-x)/\sqrt{\varepsilon}} + e^{-x/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} - \cos^2(\pi x). \tag{5.2}$$

This problem is solved using our proposed third order and fourth order convergent methods given by equations (2.13)-(2.14) with different values of h, ($h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$) and ε ($\varepsilon = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}$). The obtained maximum absolute errors are tabulated in Tables 1-10.

Example 2. Let us consider the SPBVPs:

$$\varepsilon y^{(2)} + (1 + x)y = -40(x(x^2 - 1) - 2\varepsilon), y(0) = \ y(1) = 0. \tag{5.3}$$

The analytical solution of (5.3) is

$$y(x) = 40x(1 - x). \tag{5.4}$$

This problem is solved using our proposed third order method given by equation (2.13) with $h = 2^{-4}$ and of different values of ε ($\varepsilon = 10^{-4}, 10^{-5}, 10^{-6},$

h	$\varepsilon=\frac{1}{16}$	$\varepsilon=\frac{1}{32}$	$\varepsilon=\frac{1}{64}$	$\varepsilon=\frac{1}{128}$	$\varepsilon=\frac{1}{256}$
2^{-4}	4.07-5 *	2.00-5	5.45-5	1.83-4	3.53-4
2^{-5}	2.53-6	1.24-6	3.42-6	1.22-5	3.65-5
2^{-6}	1.58-7	7.74-8	2.14-7	7.68-7	2.76-6
2^{-7}	9.87-9	4.83-9	1.34-8	4.81-8	1.83-7

Table 1: The observed maximum errors for Example 1 (fourth order given by equation (2.14); * 4.07-5= 4.07x10 $^{-5}$

h	$\varepsilon=\frac{1}{16}$	$\varepsilon=\frac{1}{32}$	$\varepsilon=\frac{1}{64}$	$\varepsilon=\frac{1}{128}$	$\varepsilon=\frac{1}{256}$
2^{-4}	4.0109-5	2.1890-5	6.7515-5	3.2952-4	1.3553-3
2^{-5}	2.5433-6	1.3229-6	2.8929-6	1.5437-5	7.4861-5
2^{-6}	1.5889-7	8.0368-8	1.6332-7	5.8818-7	3.0911-6
2^{-7}	9.9105-9	4.9357-9	1.16830-8	3.9254-8	1.1084-7

Table 2: The observed maximum errors for Example 1 (third order given by equation (2.13))

$\varepsilon=2^{-10}$		
h	Our Fourth order given by equation (2.14)	Chandra et al [1]
2^{-6}	2.637-5	2.471-5
2^{-7}	2.748-6	5.898-6
2^{-8}	1.826-7	1.211-6
2^{-9}	1.162-8	1.476-7

Table 3: The observed maximum errors for Example 1

$10^{-7}, 10^{-8}$). The obtained maximum absolute errors are tabulated in Tables 11, 12.

Two problems are considered and the numerical results of our new methods are listed in Tables 1-10 and 11, 12. These tables confirm that the obtained analysis for theoretical convergence and also illustrate the high accuracy obtained compared with other existing methods such as optimal cubic B-spline [1], cubic B-spline [5], spline function [3] and other finite difference methods [7], [8], [14]-[16].

h	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$
2^{-4}	7.09-3	4.07-3	6.30-2
2^{-5}	1.77-3	1.01-3	2.44-3
2^{-6}	4.45-4	2.54-4	9.33-4
2^{-7}	1.11-4	6.35-5	2.32-4

Table 4: The observed maximum errors for Example 1, see Kadalbajoo et al [5]

h	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{32}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$
2^{-4}	4.07-5	2.00-5	5.45-5	1.83-4
2^{-5}	2.53-6	1.24-6	3.42-6	1.22-5
2^{-6}	1.58-7	7.74-8	2.14-7	7.68-7
2^{-7}	9.87-9	4.83-9	1.34-8	4.81-8

Table 5: The observed maximum errors for Example 1, see Rashidinia et al [13]

h	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{32}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$
2^{-4}	4.07-5	2.00-5	5.45-5	1.83-4
2^{-5}	2.53-6	1.24-6	3.42-6	1.22-5
2^{-6}	1.58-7	7.74-8	2.14-7	7.68-7
2^{-7}	9.87-9	4.83-9	1.34-8	4.81-8

Table 6: The observed maximum errors for Example 1, see Tariq Aziz et al [17]

h	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{32}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$
2^{-4}	8.06-3	7.11-3	6.58-3	6.36-3
2^{-5}	2.02-3	1.79-3	1.66-3	1.61-3
2^{-6}	5.08-4	4.48-4	4.15-4	4.03-4
2^{-7}	1.27-4	1.12-4	1.04-4	1.01-4

Table 7: The observed maximum errors for Example 1, see Surla and Stojanovic [14]

h	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{32}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$
2^{-4}	4.14-3	3.68-3	3.45-3	3.45-3
2^{-5}	1.02-3	9.03-4	8.40-3	8.21-4
2^{-6}	2.54-4	5.61-5	2.08-4	2.03-4
2^{-7}	6.35-5	1.40-5	5.20-5	5.06-5

Table 8: The observed maximum errors for Example 1, see Surla and Herceg and Cvekovic [15]

h	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{32}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$
2^{-4}	1.20-4	1.28-4	1.60-4	2.34-4
2^{-5}	7.47-6	8.00-6	1.00-6	1.47-6
2^{-6}	4.67-7	5.00-7	6.26-7	9.23-7
2^{-7}	2.90-8	3.14-8	3.92-8	5.77-8

Table 9: The observed maximum errors for Example 1, see Surla and Vukoslavcevic [16]

h	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{32}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$
2^{-4}	7.09-3	5.68-3	4.07-3	6.97-3
2^{-5}	1.77-3	1.42-3	1.01-3	1.75-3
2^{-6}	4.45-4	3.55-4	2.54-4	4.33-4
2^{-7}	1.11-4	8.89-5	6.35-5	1.08-4

Table 10: The observed maximum errors for Example 1, see Kalbajoo and Bawa [3]

h	$\varepsilon = \frac{1}{16}$	$\varepsilon = \frac{1}{32}$	$\varepsilon = \frac{1}{64}$	$\varepsilon = \frac{1}{128}$	$\varepsilon = \frac{1}{256}$
2^{-4}	1.776-15	1.776-15	3.552-15	3.552-15	3.552-15
2^{-5}	4.440-15	6.217-15	1.243-14	5.329-15	5.329-15
2^{-6}	2.486-14	3.197-14	7.105-15	5.329-15	5.329-15

Table 11: The observed maximum errors for Example 2 (third order given by equation (2.13))

$h = 2^{-4}$				
ε	Miller [6]	Niijima [7]	Niijima [8]	Our Fourth order by equation (2.14)).
10^{-4}	2.5-2	2.6-2	6.5-5	7.105-15
10^{-5}	2.1-2	2.4-2	3.6-5	5.329-15
10^{-6}	7.0-3	1.7-2	3.3-5	3.552-15
10^{-7}	7.5-4	6.9-3	2.6-5	1.776-15
10^{-8}	7.4-5	2.3-3	2.0-5	3.552-15

Table 12: The observed maximum errors for Example 2

6. Conclusion

In this paper, quadratic nonpolynomial spline functions are used to develop a class of numerical methods for solving SPBVPs. These methods are computationally efficient and easy to implement on the computer. These nonpolynomial spline functions need less coefficients evaluations over other methods. Third order and fourth order convergent methods are obtained. The numerical results shown in the tables reveal that our methods give better accuracy compared with other existing methods.

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