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NEW GENERALIZATION OF PERTURBED TRAPEZOID, MID-POINT INEQUALITIES AND APPLICATIONS

Wen-Jun Liu[1 §], Qiao-Ling Xue[2], Jian-Wei Dong[3]
[1,2]College of Mathematics and Physics
Nanjing University of Information Science and Technology
Nanjing, 210044, P.R. CHINA
[1]e-mail: wjliu@nuist.edu.cn
[2]e-mail: qlx_1@yahoo.com.cn
[3]Department of Mathematics and Physics
Zhengzhou Institute of Aeronautical Industry Management
Zhengzhou, 450015, P.R. CHINA
e-mail: dongjianweiccm@163.com

Abstract: Uniform proofs, improvement of perturbed trapezoid and mid-point inequalities are established. Applications in numerical integration are also given.

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1. Introduction

Recently, N. Ujević [9] obtained the following perturbed mid-point and trapezoid inequalities

$$\left|\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)(b-a) - \frac{C}{24}(b-a)^3\right| \le \frac{\|f''-C\|_1}{8}(b-a)^2, \tag{1}$$

$$\left|\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{C}{12}(b-a)^3\right| \le \frac{\|f''-C\|_1}{8}(b-a)^2, \tag{2}$$

by defining

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§Correspondence author

$$p(t) = \begin{cases} \frac{1}{2}(t-a)^2, & t \in [a, \frac{a+b}{2}], \\ \frac{1}{2}(t-b)^2, & t \in (\frac{a+b}{2}, b], \end{cases} \quad \text{and} \quad q(t) = \frac{1}{2}(t-a)(b-t),$$

respectively, where $f : [a, b] \to R$ is such that $f' : [a, b] \to R$ is an absolutely continuous function and C is a constant. Other perturbations of the above mentioned inequalities have been examined in the past, for example see [2], [3], [4], [5], [8] and [10]. In this paper, we generalize the perturbed mid-point and trapezoid inequality by defining a uniform $p(t)$ as in (4). Our result in special case yield Theorem 3 and Theorem 7 in [9]. Finally, we give applications in numerical integration.

2. Main Results

Theorem 1. *Let $I \subset R$ be an open interval, $a, b \in I, a < b$ and let C be a constant. If $f : I \to R$ is such that $f' : [a, b] \to R$ is an absolutely continuous function. Then we have*

$$\left| \int_a^b f(t)dt - (b-a)\left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2}\right] - C\frac{1-3\lambda}{24}(b-a)^3 \right|$$
$$\leq \begin{cases} \frac{\|f''-C\|_1}{8}(1-2\lambda)(b-a)^2, & 0 \leq \lambda \leq \sqrt{2}-1, \\ \frac{\|f''-C\|_1}{8}\lambda^2(b-a)^2, & \sqrt{2}-1 < \lambda \leq 1. \end{cases} \tag{3}$$

Proof. Let $p : [a, b] \to \mathbf{R}$ be given by

$$p(t) = \begin{cases} \frac{1}{2}(t-a)[t-(1-\lambda)a-\lambda b], & t \in [a, \frac{a+b}{2}], \\ \frac{1}{2}(b-t)[\lambda a+(1-\lambda)b-t], & t \in (\frac{a+b}{2}, b]. \end{cases} \tag{4}$$

Then we have

$$\int_a^b p(t)dt = \frac{1-3\lambda}{24}(b-a)^3. \tag{5}$$

Integrating by parts, we obtain

$$\int_a^b p(t)f''(t)dt = \int_a^b f(t)dt - (b-a)\left[(1-\lambda) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2}\right]. \tag{6}$$

If $C \in R$ is a constant then from (5) and (6) it follows

$$\int_a^b p(t)[f''(t)-C]dt$$
$$= \int_a^b f(t)dt - (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right] - C\frac{1-3\lambda}{24}(b-a)^3. \quad (7)$$

We also have

$$\int_a^b p(t)[f''(t)-C]dt \le \max_{t\in[a,b]} |p(t)| \int_a^b |f''(t)-C|dt$$
$$= \begin{cases} \frac{\|f''-C\|_1}{8}(1-2\lambda)(b-a)^2, & 0 \le \lambda \le \sqrt{2}-1, \\ \frac{\|f''-C\|_1}{8}\lambda^2(b-a)^2, & \sqrt{2}-1 < \lambda \le 1. \end{cases} \quad (8)$$

From (7) and (8) we see that (3) holds. □

Remark 2. We note that in the special cases, if we take $\lambda = 0$ and $\lambda = 1$ in Theorem 1 respectively, we get Theorem 3 and Theorem 7 in [9].

Corollary 3. *Under the assumptions of Theorem 1 and with $\lambda = \frac{1}{2}$, we have the inequality*

$$\left|\int_a^b f(t)dt - \frac{1}{2}f\left(\frac{a+b}{2}\right)(b-a) - \frac{1}{2}\frac{f(a)+f(b)}{2}(b-a) + \frac{C}{48}(b-a)^3\right| \le \frac{\|f''-C\|_1}{32}(b-a)^2. \quad (9)$$

Corollary 4. *Under the assumptions of Theorem 1 and with $\lambda = \frac{2}{3}$, we have the inequality*

$$\left|\int_a^b f(t)dt - \frac{1}{3}f\left(\frac{a+b}{2}\right)(b-a) - \frac{2}{3}\frac{f(a)+f(b)}{2}(b-a) + \frac{C}{24}(b-a)^3\right| \le \frac{\|f''-C\|_1}{18}(b-a)^2. \quad (10)$$

Corollary 5. *Under the assumptions of Theorem 1 and with $\lambda = \frac{1}{3}$, we have the Simpson inequality*

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{\|f''-C\|_1}{24}(b-a)^2. \quad (11)$$

Theorem 6. *Under the assumptions of Theorem 1 suppose that there exist constants $\gamma, \Gamma \in R, \gamma \le f''(t) \le \Gamma, t \in [a,b]$. Then*

$$\left|\int_a^b f(t)dt - (b-a)\left[(1-\lambda)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2}\right]\right.$$

$$-\gamma\frac{1-3\lambda}{24}(b-a)^3\Bigg| \le \begin{cases} \frac{S-\gamma}{8}(1-2\lambda)(b-a)^3, & 0\le\lambda\le\sqrt{2}-1, \\ \frac{S-\gamma}{8}\lambda^2(b-a)^3, & \sqrt{2}-1<\lambda\le 1, \end{cases} \tag{12}$$

where $S = \dfrac{f'(b)-f'(a)}{b-a}$.

Proof. Choose $C=\gamma$ in Theorem 1 and utilize $\|f''-C\|_1=(S-\gamma)(b-a)$ we can prove that (12) holds. □

Remark 7. We note that in the special cases, if we take $\lambda = 0$ and $\lambda = 1$ in Theorem 6 respectively, we get Corollary 4 and Corollary 8 in [9].

Theorem 8. *Let the assumptions of Theorem 1 be satisfied. Then we have*

$$\Bigg|\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) + \frac{C}{12}(b-a)\{(a^2+4ab+b^2)-6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\}\Bigg| \le \begin{cases} \frac{\|f''-C\|_1}{8}(1-2\lambda)^2(b-a)^2, & \lambda\in\left[0,\frac{1}{2}-\frac{\sqrt{2}}{4}\right]\cup\left(\frac{1}{2}+\frac{\sqrt{2}}{4},1\right], \\ \frac{\|f''-C\|_1}{2}\lambda(1-\lambda)(b-a)^2, & \lambda\in\left(\frac{1}{2}-\frac{\sqrt{2}}{4},\frac{1}{2}+\frac{\sqrt{2}}{4}\right]. \end{cases} \tag{13}$$

Proof. Let $q:[a,b]\to\mathbf{R}$ be given by

$$q(t)=\frac{1}{2}[t-(1-\lambda)a-\lambda b][\lambda a+(1-\lambda)b-t], \tag{14}$$

we have

$$\int_a^b q(t)dt = \frac{1}{12}(b-a)\{(a^2+4ab+b^2)-6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\}. \tag{15}$$

Integrating by parts, we obtain

$$\int_a^b q(t)f''(t)dt = \frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)dt. \tag{16}$$

From (15) and (16) it follows

$$\int_a^b q(t)[f''(t)-C]dt = -\int_a^b f(t)dt + \frac{f(a)+f(b)}{2}(b-a) - \frac{C}{12}(b-a)\{(a^2+4ab+b^2)-6[(1-\lambda)a+\lambda b][\lambda a+(1-\lambda)b]\}. \tag{17}$$

We also have

$$\int_a^b q(t)[f''(t)-C]dt \le \max_{t\in[a,b]}|q(t)|\int_a^b |f''(t)-C|dt$$

$$= \begin{cases} \frac{\|f''-C\|_1}{8}(1-2\lambda)^2(b-a)^2, & \lambda \in \left[0, \frac{1}{2}-\frac{\sqrt{2}}{4}\right] \cup \left(\frac{1}{2}+\frac{\sqrt{2}}{4}, 1\right], \\ \frac{\|f''-C\|_1}{2}\lambda(1-\lambda)(b-a)^2, & \lambda \in \left(\frac{1}{2}-\frac{\sqrt{2}}{4}, \frac{1}{2}+\frac{\sqrt{2}}{4}\right]. \end{cases} \tag{18}$$

From (17) and (18) we see that (13) holds. □

Remark 9. We note that in the special cases, if we take $\lambda = 0$ or $\lambda = 1$ in Theorem 8, we can also get Theorem 7 in [9].

Corollary 10. *Under the assumptions of Theorem 8 and with $\lambda = \frac{1}{2}$, we have another perturbed trapezoid inequality*

$$\left| \int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) - \frac{C}{24}(b-a)^3 \right| \leq \frac{\|f''-C\|_1}{8}(b-a)^2. \tag{19}$$

Theorem 11. *Under the assumptions of Theorem 8 suppose that there exist constants $\gamma, \Gamma \in R, \gamma \leq f''(t) \leq \Gamma, t \in [a,b]$. Then*

$$\begin{aligned} &\left| \int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) \right. \\ &\quad \left. + \frac{\gamma}{12}(b-a)\{(a^2+4ab+b^2) - 6[(1-\lambda)a+\lambda b][\lambda a + (1-\lambda)b]\} \right| \\ &\quad \leq \begin{cases} \frac{S-\gamma}{8}(1-2\lambda)^2(b-a)^3, & \lambda \in \left[0, \frac{1}{2}-\frac{\sqrt{2}}{4}\right] \cup \left(\frac{1}{2}+\frac{\sqrt{2}}{4}, 1\right], \\ \frac{S-\gamma}{2}\lambda(1-\lambda)(b-a)^3, & \lambda \in \left(\frac{1}{2}-\frac{\sqrt{2}}{4}, \frac{1}{2}+\frac{\sqrt{2}}{4}\right], \end{cases} \end{aligned} \tag{20}$$

where $S = \frac{f'(b)-f'(a)}{b-a}$.

Proof. Choose $C = \gamma$ in Theorem 8 and utilize $\|f''-C\|_1 = (S-\gamma)(b-a)$ we can prove that (20) holds. □

Remark 12. We note that in the special cases, if we take $\lambda = 0$ and $\lambda = 1$ in Theorem 11 respectively, we can also get Corollary 8 in [9].

3. Applications in Numerical Integration

Theorem 13. *Let the assumptions of Theorem 6 hold. If $D = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a given division of the interval $[a,b]$ then we have*

$$\int_a^b f(t)dt = A_{MT}(f,D) + R_{MT}(f,D),$$

where

$$A_{MT}(f,D) = \sum_{i=0}^{n-1} h_i \left[(1-\lambda) f\left(\frac{x_i + x_{i+1}}{2}\right) + \lambda \frac{f(x_i) + f(x_{i+1})}{2}\right] + \gamma \frac{1-3\lambda}{24} \sum_{i=0}^{n-1} h_i^3,$$

$$|R_{MT}(f,D)| \leq \begin{cases} (1-2\lambda) \sum_{i=0}^{n-1} \frac{S_i - \gamma}{8} h_i^3, & 0 \leq \lambda \leq \sqrt{2}-1, \\ \lambda^2 \sum_{i=0}^{n-1} \frac{S_i - \gamma}{8} h_i^3, & \sqrt{2}-1 < \lambda \leq 1, \end{cases}$$

and $h_i = x_{i+1} - x_i$, $S_i = \frac{f'(x_{i+1}) - f'(x_i)}{h_i}$, $i = 0, 1, 2, \cdots, n-1$.

Proof. Apply Theorem 6 to the interval $[x_i, x_{i+1}], i = 0, 1, 2, \cdots, n-1$ and sum. Then use the triangle inequality to obtain the desired result. □

Theorem 14. *Let the assumptions of Theorem 11 hold. Then we have*

$$\int_a^b f(t)dt = A_T(f,D) + R_T(f,D),$$

where

$$A_T(f,D) = \sum_{i=0}^{n-1} h_i \frac{f(x_i) + f(x_{i+1})}{2}$$

$$- \frac{\gamma}{12} \sum_{i=0}^{n-1} h_i \{(x_i^2 + 4x_i x_{i+1} + x_{i+1}^2) - 6[(1-\lambda)x_i + \lambda x_{i+1}][\lambda x_i + (1-\lambda)x_{i+1}]\},$$

$$|R_T(f,D)| \leq \begin{cases} (1-2\lambda)^2 \sum_{i=0}^{n-1} \frac{S_i - \gamma}{8} h_i^3, & \lambda \in \left[0, \frac{1}{2} - \frac{\sqrt{2}}{4}\right] \cup \left(\frac{1}{2} + \frac{\sqrt{2}}{4}, 1\right], \\ \lambda(1-\lambda) \sum_{i=0}^{n-1} \frac{S_i - \gamma}{2} h_i^3, & \lambda \in \left(\frac{1}{2} - \frac{\sqrt{2}}{4}, \frac{1}{2} + \frac{\sqrt{2}}{4}\right], \end{cases}$$

and $h_i = x_{i+1} - x_i$, $S_i = \frac{f'(x_{i+1}) - f'(x_i)}{h_i}$, $i = 0, 1, 2, \cdots, n-1$.

Proof. Apply Theorem 11 to the interval $[x_i, x_{i+1}], i = 0, 1, 2, \cdots, n-1$ and sum. Then use the triangle inequality to obtain the desired result. □

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