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ONE-WAY ANALYSIS OF VARIANCE UNDER HETEROSCEDASTICITY FOR TESTING THE EQUIVALENCY OF MEANS AGAINST AN ORDERED ALTERNATIVE WITH VARYING INITIAL SAMPLE SIZES

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Abstract: The conventional F test of *ANOVA*, is only applicable when the assumption of the equal variances holds. The departures from the equal variances mislead authors to wrong conclusions, especially if the sample sizes are not equal. There appear many modifications of the statistic F in literature. These modifications mostly cover the employability of it in case of unequal variances. There after approximate distributions are derived. In this study we use a general one-stage method described by Chen et al [6] for testing the equality of normal means against an ordered alternative in one-way layout, when variances are unknown and unequal. By simulation, this method is applied to randomly chosen $n_0 = n_i - 1$ and n_0 $(2 \leq n_0 < n_i)$ of n_i observations. Then the calculated powers and the approximations of the reject ratios of a true null hypothesis to the given level α are compared, with each other.

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1. Introduction

There exist several numbers of approximate and exact tests for comparing two normal means, under equal variances. But there are only a few for testing equal means of more than two populations, of which the population variances are quite different [11]. Chochran [8] proposed a test statistic, where the inverses of the sample variances were used as weights and its distribution converges to chi-square. Later, Welch [12] developed the statistic W, having an appoximate distribution F using the sample sizes divided by the sample variances as weights. The problem of testing ordered normal means with equal variances has been considered by several authors. Bartholomew [1]-[4] constructed the likelihood ratio test using maximum likelihood estimates of the normal means under restriction. A test procedure for identifying the lowest dose in which μ_1 corresponds to a control and $\mu_2, \ldots, \mu_k$ to increasing doses of a drug was derived by Williams [14]. Furthermore, Williams [15] gave a limiting distribution of the test for hypothesis concerning monotonically ordered normal means. When the variances are unknown and unequal, Marcus [10] gave a test procedure for testing homogenity of means against ordered alternatives in the analysis of variance model by using a two-stage procedure proposed by Bishop and Dudewics [5]. Wen and Chen [13] employed the one-stage procedure, developed by Chen and Lam, for interval estimation of the largest normal mean. Chen et al used the one-stage procedure described by Chen for testing the equality of normal means against an ordered alternative in one-way layout when the variances are unknown and unequal [6, 7]. They proposed a general one-stage procedure based on a statistic $\widetilde{R}$. In this article we applied this procedure to two initial sample sizes. Then the powers of the test and the approximations of the reject ratios of a true null hypothesis to the given level α are compared with each other.

2. The General One-Stage Sampling Procedure

Let Y_{ij} $(j = 1, \ldots, n_i)$ be an independent random sample of size $n_i\,(\geq 3)$ from a normal population π_i $(i = 1, \ldots, I)$ with unknown mean μ_i and unknown variance σ_i^2. For each population, use the first (or any randomly chosen) n_0 $(2 \leq n_0 < n_i)$ observations to calculate the usual sample mean and sample vari-

			α			
			0.10	0.05	0.025	0.01
I	n_i	n_0	$\mu_i = 10, 6, 2$ $\sigma_i^2 = 1, 1.5, 2$			
3	11	10	0.8030	0.6796	0.5321	0.3652
		4	0.7472	0.5810	0,3903	0.1759
	21	20	0.9106	0.8399	0.7444	0.6088
		10	0.8790	0.7917	0.6718	0.5163
	31	30	0.9138	0.8521	0.7685	0.6406
		15	0.8997	0.8165	0.7202	0.5764
$\mu_i = 12, 10, 8, 6, 4, 2$ $\sigma_i^2 = 1, 1.3, 1.8, 2.3, 1.8, 3.3$						
6	11	10	0.7855	0.6491	0.4973	0.3168
		4	0.5992	0.3905	0.2079	0.0682
	21	20	0.8145	0.6984	0.5663	0.4059
		10	0.7820	0.6468	0.5029	0.3440
	31	30	0.8363	0.7205	0.5882	0.4320
		15	0.8168	0.6864	0.5587	0.3997

Table 1: The values of power, when $Y_i \sim N\left(\mu_i, \sigma_i^2\right)$, $n_i = [11, 21, 31]$, $n_0 = [10, 4]$, $[20, 10]$, $[30, 15]$ and $I = 3, 6$

ance, respectively, by

$$\overline{Y_i} = \sum_{j=1}^{n_0} Y_{ij}/n_0$$

and

$$S_i^2 = \sum_{j=1}^{n_0} \left(Y_{ij} - \overline{Y_i}\right)^2 / n_0 - 1.$$

Define the weights u_i and v_i for the observations in the i-th sample as

$$u_i = \frac{1}{n_i} + \frac{1}{n_i}\sqrt{\frac{n_i - n_0}{n_0}\left(n_i Z^*/S_i^2 - 1\right)},$$

$$v_i = \frac{1}{n_i} - \frac{1}{n_i}\sqrt{\frac{n_0}{n_i - n_0}\left(n_i Z^*/S_i^2 - 1\right)},$$

where Z^* is the maximum of $\left\{S_i^2/n_1, \ldots, S_I^2/n_I\right\}$, see [6]. The final weighted sample mean using all observations is calculated from

$$\widetilde{Y_{i.}} = \sum_{j=1}^{n_0} u_i Y_{ij} + \sum_{j=n_{0+1}}^{n_i} v_i Y_{ij}. \tag{1}$$

			α			
			0.10	0.05	0.025	0.01
I	n_i	n_0	$\mu_i = 0 \quad \sigma_i^2 = 1, 1.3, 1.8$			
3	11	10	0.0991	0.0533	0.0295	0.0111
		4	0.0984	0.0466	0.0272	0.0095
	21	20	0.1016	0.0500	0.0294	0.0096
		10	0.0999	0.0488	0.0231	0.0084
	31	30	0.0997	0.0482	0.0259	0.0087
		15	0.1047	0.0513	0.0246	0.0096
			$\mu_i = 0 \quad \sigma_i^2 = 1, 1.3, 1.8, 2.3, 1.8, 3.3$			
6	11	10	0.0996	0.0480	0.0238	0.0096
		4	0.0907	0.0462	0.0227	0.0090
	21	20	0.1015	0.0520	0.0237	0.0096
		10	0.0921	0.0450	0.0215	0.0092
	31	30	0.0988	0.0496	0.0277	0.0087
		15	0.0987	0.0493	0.0228	0.0077

Table 2: The values of the estimated reject ratios, when $Y_i \sim N\left(0, \sigma_i^2\right)$, $n_i = [11, 21, 31]$, $n_0 = [10, 4]$, $[20, 10]\,, [30, 15]$ and $I = 3, 6$

Given the sample variances S_i^2 $(i = 1, \ldots, I)$, the weighed sample mean $\widetilde{Y_{i.}}$ has a conditional normal distribution and the transformations

$$t_i = \frac{\widetilde{Y_{i.}} - \mu_i}{\sqrt{Z^*}}\,, \quad i = 1, \ldots, I\,,$$

have i.i.d. t distributions with $v = n_0 - 1$ degrees of freedom [6]. The model, considered for the one-way layout, is given by

$$Y_{ij} = \mu + \alpha_i + e_{ij}\,, \quad i = 1, \ldots, I\,, \quad j = 1, \ldots, n_i,$$

where $\sum_{i=1}^{I} \alpha_i = 0$ and the e_{ij}'s are independent random variables with $e_{ij} \sim N\left(0, \sigma_i^2\right)$.μ_i denotes $\mu + \alpha_i$. The hypothesis that the population means (μ_i's) are all equal against the ordered alternative, i.e.,

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_I \text{ vs. } H_a : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_I,$$

with at least one strict inequality, is the hypothesis planned for testing. Let us assume that the one-stage sampling procedure has been constructed. Then the final weighted sample means $\widetilde{Y_{i.}}$ have been calculated from equation (1).

Thereafter, $\widetilde{U}$ and $\widetilde{V}$ have been defined as follows [6]:

$$\widetilde{U} = \max_{1\le r\le I} \frac{1}{r}\sum_{i=1}^{r}\frac{\widetilde{Y_{i.}}}{\sqrt{Z^*}}\,, \tag{2}$$

$$\widetilde{V} = \min_{1\le r\le I} \frac{1}{I-r+1}\sum_{i=r}^{I}\frac{\widetilde{Y_{i.}}}{\sqrt{Z^*}}\,. \tag{3}$$

A test statistic for testing H_0 against H_a is given as

$$\widetilde{R} = \widetilde{U} - \widetilde{V}.$$

Using equation (2) and equation (3) $\widetilde{R}$ can be written as

$$\widetilde{R} = \max_{1\le r\le I} \frac{1}{r}\sum_{i=1}^{r}\frac{\widetilde{Y_{i.}} - \mu_i + \mu_i}{\sqrt{Z^*}} - \min_{1\le r\le I} \frac{1}{I-r+1}\sum_{i=r}^{I}\frac{\widetilde{Y_{i.}} - \mu_i + \mu_i}{\sqrt{Z^*}}\,,$$

$$\widetilde{R} = \max_{1\le r\le I} \frac{1}{r}\sum_{i=1}^{r}\frac{t_i + \mu_i}{\sqrt{Z^*}} - \min_{1\le r\le I} \frac{1}{I-r+1}\sum_{i=r}^{I}\frac{t_i + \mu_i}{\sqrt{Z^*}},$$

where t_i's are i.i.d. t random variables with v d.f. Under the null hypothesis, the distribution of the test statistic $\widetilde{R}$ becomes

$$\widetilde{R} = \max_{1\le r\le I} \frac{1}{r}\sum_{i=1}^{r} t_i - \min_{1\le r\le I} \frac{1}{I-r+1}\sum_{i=1}^{r} t_i\,.$$

The hypothesis H_0 is rejected if

$$\widetilde{R} > q_{\alpha,I,v}\,,$$

where $q_{\alpha,I,v}$ is the upper α percentage point of the distribution of $\widetilde{Q}$. Chen et al [6] obtained the critical values by Monte Carlo simulations for various combinations of $I = 3, 4, 6, 10$, degrees of freedom $v = 3, 5, 9, 14, 19, 24, 29, 59$ and level of significance $\alpha = 0.10, 0.05, 0.025, 0.01$.

3. Simulation

The general one-stage test is applied to two different initial sample sizes. These are $n_0 = n_i - 1$ and $2 \le n_0 < n_i$. The data is generated 10000 times from the distributions $Y_i \sim N\left(\mu_i, \sigma_i^2\right)$, for $i = 1, \dots I$ and $I = 3, 6$. We took sample sizes (n_i) 11, 21, 31 with initial sample sizes (n_0) as [10, 4], [20, 10], [30, 15]. In each simulation run $\widetilde{R}$'s are calculated and compared with the critical values given by Chen et al [6]. Finally, the average values of the reject ratios of the null hypothesis under unequal means are reported in Table 1. The simulation results reveal that if $n_0 = 10, 20, 30$, the test procedure gave the best power for the given level of significances $\alpha = .10, .05, .025, .01$. But when it is $4, 10, 15$ the test procedure is not so good as it is $10, 20, 30$.

Furthermore, this test is again applied to the same initial samples. But this time, the appoximations of the reject ratios for the true null hypothesis to the level α's are taken into consideration. The data is generated from the distributions $Y_i \sim N\left(0, \sigma_i^2\right)$ and $\widetilde{R}$'s are calculated under equal means. The average values of the reject ratios for the true null hypothesis are summarized in Table 2. From the simulation results, we found out that if n_0 is $4, 10, 15$ and $I = 3$, the average values of the reject ratios for the test are closer to the level α's compared with the average values of the reject ratios for the test if $n_0 = 10, 20, 30$. On the other side, these values are closer to α's, when $n_0 = 10, 20, 30$ and $I = 6$. Finally, the general one-stage test with small initial sample sizes n_0, might be an appropriate choise for a researcher, when the number of population is 3. In addition, nearly half of the estimated values underestimate the level α when $I = 3$, but most of the estimated values underestimate it when $I = 6$.

4. Discussion and Conclusion

For comparing normal populations under the assumption of equal variances, it is known that the F test of *ANOVA* is the best for all levels of sample sizes and number of groups [9]. In the minor departures from the equal variance assumption, the general one-stage method is applicable for testing homogeneity of means against an ordered alternative. In this study two different initial sample sizes are used to see how the power and the appoximations of the reject ratios to the given level of significance α differ. The conclusion is that when n_0 is the first drawn $n_i - 1$ observation, the one-stage method performs better. The results in Table 2 show that initial sample size will not be any problem if $I = 3$. However, if $I = 6$, it is better to use the initial sample size (n_0) as $n_i - 1$, because of the approximations of the estimated reject ratios to the level of significances α's. Finally, under unequal variances, we recommend that the researcher may use the general one-stage method with initial sample size $n_0 = n_i - 1$, especially when the number of the populations is large.

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