International Journal of Pure and Applied Mathematics
Volume 41 No. 6 2007, 793-796

ON RADICAL-INJECTIVE MODULES

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Abstract: In this paper, we give a definition of radical-injective modules, then give some properties and an equivalent characterization of radical-injective modules. For example, it is shown that a radical-injective module is injective module if and only if it has an injective envelope. Finally, the endomorphism ring of radical-injective module is discussed.

AMS Subject Classification: 26A33
Key Words: injective module, radical-injective module, injective envelope, endomorphism ring

1. Introduction

Let R be a ring with Jacobson radical $J(R)$, Given a left R-module M, A Submodule K of M is said to be essential (written $K \trianglelefteq M$), for every submodule L of M, if $K \cap L = 0$ implies $L = 0$. A homomorphism $f : K \longrightarrow M$ in $R-Mod$ is called an essential monomorphism if it is a monomorphism with $\mathrm{Im} f \trianglelefteq M$.

Radical-projective modules were introduced and studied in [5] and [2]. Now, we will difine the radical-injective module and give some properties of it. A left R-module M is called radical-injective module, if for every monomorphism $g : A \longrightarrow B$ in the left R-module category and every homomorphism $f : A \longrightarrow M$, there exists $h : B \longrightarrow M$, such that $\ker(f - hg) \trianglelefteq A$. In this paper,

Received: June 11, 2007
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it is shown that the class of radical-injective modules is closed under finite direct products and direct product divisors (Proposition 2.1); an R-module M is radical-injective if and only if each exact sequence: $0 \longrightarrow M \longrightarrow L \longrightarrow K \longrightarrow 0$ in $R - Mod$, there exist a submodule N of L and an essential monomorphism $h : N \longrightarrow M$ (Proposition 2.2); a radical-injective module is injective if and only if it has an injective envelope (Proposition 2.3). Finally, the endomorphism ring of radical-injective modules is discussed (Theorem 3.4).

Throughout this paper, R will denote an associate ring with identity, module will mean left R-module and homomorphisms will operate on the right. As usual, $J(R)$ will stand for the Jacobson radical of R; Soc$M$ is the socle of M and $R - Mod$ will be the category whose objects are left R-modules.

2. Main Results

We have the following results about Radical-injective modules.

Proposition 1. *Suppose $\{M_i \mid i \in I\}$ be a finite set of modules. The following statements are equivalent:*

(1) *M_i is a radical-injective module for each $i \in I$.*

(2) *$\Pi_{i\in I}M_i$ is radical-injective.*

Proof. (1) $\Longrightarrow$ (2) Suppose M_i be radical-injective for each $i \in I$, $g : A \longrightarrow B$ is a monomorphism in $R - Mod$, $f : A \longrightarrow \Pi_{i\in I}M_i$ a homomorphism and $\pi_i : \Pi_{i\in I}M_i \longrightarrow M_i$ is the surjective map, for each $i \in I$. Since M_i is a radical-injective module, there exists $h_i : B \to M_i$ such that $\ker(\pi_i f - h_i g) \trianglelefteq A$, for each $i \in I$. Set $h = \Pi_{i\in I}h_i : B \longrightarrow \Pi_{i\in I}M_i$, then $\ker(f - hg) = \ker(f - \Pi_{i\in I}h_i g) = \Pi_{i\in I}\ker(\pi_i f - h_i g)$, since $\ker(\pi_i f - h_i g) \trianglelefteq A$, then $\ker(f - hg) \trianglelefteq A$, Therefore $\Pi_{i\in I}M_i$ is radical-injective.

(2) $\Longrightarrow$ (1) Suppose $\Pi_{i\in I}M_i$ is radical-injective, we have: $\ker(f - hg) \trianglelefteq A$, and g is monomorphism, let $h_i = \pi_i h$, then $\ker(\pi_i f - h_i g) = \ker(\pi_i f - \pi_i hg) = \ker \pi_i(f - hg)$. since $\ker(f - hg) \trianglelefteq A$, then $\ker \pi_i(f - hg) \trianglelefteq A$, therefore M_i is radical-injective, for each $i \in I$. □

Proposition 2. *Suppose $0 \longrightarrow M \longrightarrow L \longrightarrow K \longrightarrow 0$ be an exact sequence in $R - Mod$, where M is radical-injective. Then there exist a submodule N of L and an essential monomorphism $h : N \longrightarrow M$.*

Proof. Since M is a radical-injective $R - Module$, for the identical morphism

$1_M : M \to M$, there exists $h : L \longrightarrow M$, such that $\ker(1_M - hg) \trianglelefteq M$, Set $N =$ Img, then N is the a submodule of L, it is enough to show that $h : N \longrightarrow M$ is essential. Consider the composition homomorphism $hg : M \longrightarrow L \longrightarrow M$. Since $0 = \ker(1_M - hg) \cap \ker hg$ and $\ker(1_M - hg) \trianglelefteq M$, we have: $\ker hg = 0$. Note that $\ker(1_M - hg) \subseteq \mathrm{Im}(h \mid N)$ and $\ker(1_M - hg) \trianglelefteq M$, which implies that $\mathrm{Im}(h \mid N) \trianglelefteq M$, Thus $\mathrm{Im}(h \mid N)$ is essential in M, and so the monomorphism $h : N \longrightarrow M$ is essential. □

Definition 3. A pair (E, i) is an injective envelope of M in case E is an injective left R-module and $0 \longrightarrow M \longrightarrow E$ is an essential monomorphism.

Theorem 4. *Suppose $M \in R - Mod$ is radical-injective, then M is an injective module if and only if it has an injective envelope $i : M \longrightarrow E$.*

Proof. First we prove the sufficiency. By the radical-injective of M, for the identical morphism $1_M : M \to M$, there exists $f : E \to M$, such that $\ker(1_M - fi) \trianglelefteq M$. Since $0 = \ker(1_M - fi) \cap \ker fi$, we have $\ker fi = 0$, so fi is monomorphism. It follows that: $\mathrm{Im}i \cap \ker f = 0$, and hence $\ker f = 0$ by $i : M \longrightarrow E$ is essential, so f is monomorphism. Since E is injective, the monomorphism $f : E \longrightarrow M$ splits, i.e., there is an epimorphism $g : M \longrightarrow E$ such that $gf = 1_E$. Hence $M = \ker g \oplus \mathrm{Im}f$, Since f is monomorphism, then $\mathrm{Im}f \cong E$. but $\ker(1_M - fi) \leq \mathrm{Im}f$ and $\ker(1_M - fi) \trianglelefteq M$, so $\mathrm{Im}f \trianglelefteq M$, thus $\ker g = 0$ by $\ker g \cap \mathrm{Im}f = 0$, so $M = \mathrm{Im}f \cong E$, and E is injective, therefore M is injective. Since M is injective, and hence M is the injective envelope of M, then the necessity is obvious. □

Definition 5. For any module M, the sum of all simple submodules of M is called the socle of M, denoted $\mathrm{Soc}(M)$.

Lemma 6. (see [1]) *If M is a left $R-module$, then $\mathrm{Soc}M = \sum\{K \leq M | K$ is minimal in $M\} = \bigcap\{L \leq M | L$ is essential in $M\}$.*

Theorem 7. *Suppose P be a radical-injective $R-module$, $S = \mathrm{Hom}_R(P, P)$ the ring of endomorphism of P, and $T = \{f \in S$: $\ker f$ is essential in $P\}$. Then $J(S) \subseteq T \subseteq \mathrm{Hom}_R(P/\mathrm{Soc}P, P)$*

Proof. Let $f \in J(S)$ and suppose each $K \leq P$, $\ker f \cap K = 0$. It is enough to show that: $K = 0$. Let $l : K \longrightarrow P$ be the natural embed. Then $\ker f \cap K = 0$ implies that $fl : K \longrightarrow P$ is an monomorphism. Since P is radical-injective, there is an $R-$ homomorphism $g : P \longrightarrow P$ such that $\ker(l - gfl) \trianglelefteq K$, since $f \in J(S)$, then $1_P - gf$ is an isomorphism on P. Thus $0 = \ker(1_P - gf)l = \ker(l - gfl) \trianglelefteq K$, so $K = 0$. It follows that $\ker f$ is essential, and hence $f \in T$. By Lemma 6 (see [1]) we know that Soc$P$ can be contained in all essential submodules of P. Thus if $f \in T$, then $\mathrm{Soc}P \subseteq \ker f$, and hence f can be thought of as a homomorphism from $P/\mathrm{Soc}P \longrightarrow P$, i.e. $f \in \mathrm{Hom}_R(P/\mathrm{Soc}P, P)$, therefore $J(S) \subseteq T \subseteq \mathrm{Hom}_R(P/\mathrm{Soc}P, P)$. □

Acknowledgements

This research was supported by the Natural Science Foundation of Education Department of Anhui Province under grant 2006KJ050C.

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