International Journal of Pure and Applied Mathematics
Volume 41 No. 6 2007, 797-805

ON INTEGRAL MEAN ESTIMATES FOR POLYNOMIALS

Barchand Chanam[1], K.K. Dewan[2 §]

[1,2]Department of Mathematics
Faculty of Natural Sciences
Jamia Millia Islamia – Central University
New Delhi, 110025, INDIA

Abstract: Let $p(z)$ be a polynomial of degree n having all its zeros in $|z| \le k$, where $k \ge 1$, then it is known that for each $r \ge 1$,

$$n\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} |1+k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1} |p'(z)| .$$

In this paper, we first obtain an improvement as well as an extension for $r > 0$, of the above inequality by considering the class of polynomials of degree $n \ge 3$.

AMS Subject Classification: 30A10, 30C10, 30C15
Key Words: polynomials, L^r inequalities, maximum modulus

1. Introduction and Statement of Results

Let $p(z)$ be a polynomial of degree n. It was Turán [10] who showed that if $p(z)$ has all its zeros in $|z| \le 1$, then

$$n \max_{|z|=1} |p(z)| \le 2 \max_{|z|=1} |p'(z)| . \tag{1.1}$$

Inequality (1.1) is best possible with equality for $p(z) = az^n + \beta$, where $|\alpha| = |\beta|$.

As an extension of (1.1), Govil [6] (see also [2]) proved that if $p(z)$ has all its zeros in $|z| \le k$, where $k \ge 1$, then

$$n \max_{|z|=1} |p(z)| \le (1+k^n) \max_{|z|=1} |p'(z)| . \tag{1.2}$$

Here equality occurs for $p(z) = \alpha z^n + \beta k^n$, where $|\alpha| = |\beta|$ and $k \ge 1$.

Received: June 16, 2007

§Correspondence author

Malik [8] obtained a generalization of (1.1) in the sense that left-hand side of (1.1) is replaced by a factor involving the integral mean of $|p(z)|$ on $|z| = 1$. In fact, he proved that if $p(z)$ has all its zeros in $|z| \leq 1$, then for each $r > 0$,

$$n\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1+e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1} |p'(z)|\,. \tag{1.3}$$

The corresponding generalization of (1.2) which is an extension of (1.3) was obtained by Aziz [3] by proving the following result.

Theorem A. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, where $k \geq 1$, then for each $r \geq 1$,*

$$n\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1+k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1} |p'(z)|\,. \tag{1.4}$$

The result is best possible and equality in (1.4) occurs for the polynomial $p(z) = \alpha z^n + \beta k^n$, where $|\alpha| = |\beta|$.

In this paper, we first obtain an improvement as well as an extension for $r > 0$ of (1.4) by considering the class of polynomials of degree $n \geq 3$. More precisely, we prove

Theorem 1. *If $p(z) = \sum\limits_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, where $k \geq 1$, then for each $r > 0$,*

$$\begin{aligned} n\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} & \\ \leq \left\{\int_0^{2\pi} |1+k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} & \left[\max_{|z|=1} |p'(z)| - \frac{2|a_1|}{n+1}\left(1-\frac{1}{k^{n-1}}\right)\right. \\ & \left. - \frac{2|a_2|}{k^{n-1}}\left\{\frac{(k^{n-1}-1)}{n-1} - \frac{(k^{n-3}-1)}{n-3}\right\}\right] \quad \text{for } n > 3, \end{aligned}$$

and

$$\begin{aligned} 3\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \leq & \left\{\int_0^{2\pi} |1+k^3 e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \\ & \times\left[\max_{|z|=1} |p'(z)| - \frac{(k-1)}{2k^2}\{(k+1)|a_1| + (k-1)2|a_2|\}\right] \text{ for } n > 3. \end{aligned}$$

The result is best possible and the extremal polynomial is $p(z) = \alpha z^n + \beta k^n$, where $|\alpha| = |\beta|$.

Remark 1. For $k = 1$, Theorem 1 reduces to (1.3) for $n \geq 3$.

Remark 2. For $k > 1$, $\frac{k^x - 1}{x}$ with $x > 1$, is an increasing function of x and so the expression $\left(\frac{k^{n-1}}{n-1} - \frac{k^{n-3}-1}{n-3}\right)$ is always non-negative so that for polynomials of degree $n > 3$, (1.5) provides a refinement of Theorem A.

Further, for $n = 3$, it is clear that (1.6) gives an improvement of Theorem A as well. In fact, excepting the cases when $k = 1$, and $a_1 = 0$ and $a_2 = 0$, the bound obtained in Theorem 1 is always sharper than the bound obtained from Theorem A.

Letting $r \to \infty$ in (1.5) and (1.6), we get the following result which is an improvement of (1.2) for polynomials of degree $n \geq 3$.

Corollary 1. *If $p(z)$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, where $k \geq 1$, then*

$$n \max_{|z|=1} |p(z)| \leq (1 + k^n) \left[\max_{|z|=1} |p'(z)| - \frac{2|a_1|}{n+1}\left(1 - \frac{1}{k^{n-1}}\right) - \frac{2|a_2|}{k^{n-1}} \times \left\{\frac{(k^{n-1}-1)}{n-1} - \frac{(k^{n-3}-1)}{n-3}\right\}\right] \quad \text{for } n > 3 \tag{1.5}$$

and

$$3 \max_{|z|=1} |p(z)| \leq (1 + k^3) \left[\max_{|z|=1} |p'(z)| - \frac{(k-1)}{2k^2}\{(k+1)|a_1| + (k-1)2|a_2|\}\right], \quad \text{for } n = 3\,. \tag{1.6}$$

The result is sharp and the extremal polynomial being $p(z) = \alpha z^n + \beta k^n$, where $|\alpha| = |\beta|$.

Next, we further obtain the following refinement of Theorem 1, which gives an improvement of the result due to Aziz and Ahemad [1] for $n \geq 3$.

Theorem 2. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n \geq 3$ having all its zeros in $|z| \leq k$, where $k \geq 1$ and $m = \min_{|z|=k} |p(z)|$, then for every real or complex number α with $|\alpha| \leq 1$ and for real $r > 0$,*

$$n \left\{\int_0^{2\pi} |p(e^{i\theta}) + \alpha m|^r d\theta\right\}^{\frac{1}{r}} \leq \left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \left[\max_{|z|=1} |p'(z)| - \frac{2|a_1|}{n+1}\left(1 - \frac{1}{k^{n+1}}\right) - \frac{2|a_2|}{k^{n-1}}\left\{\frac{(k^{n-1}-1)}{n-1} - \frac{(k^{n-3}-1)}{n-3}\right\} \quad \text{for } n = 3\right.$$

and

$$3\left\{\int_0^{2\pi}|p(e^{i\theta})+\alpha m|^r d\theta\right\}^{\frac{1}{r}}$$
$$\leq\left\{\int_0^{2\pi}|1+k^3e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}\left[\max_{|z|=1}|p'(z)|-\frac{(k-1)}{2k^2}\{(k+1)|a_1|+(k-1)2|a_2|\}\right],$$
$$\text{for } n=3.$$

The result is best possible and equality in (1.9) *and* (1.10) *holds for* $p(z)=z^n+k^n$.

Letting $r\to\infty$ and $|\alpha|\to 1$ in (1.9) and (1.10) and choosing the argument of α suitably such that $|p(e^{i\theta})+\alpha m|=|p(e^{i\theta})|+|\alpha|m$, we get the following result, which is an improvement of the result due to Govil [7] for polynomials of degree $n\geq 3$.

Corollary 2. *If* $p(z)$ *is a polynomial of degree* $n\geq 3$ *having all its zeros in* $|z|\leq k$, $k\geq 1$, *then*

$$n\left[\max_{|z|=1}|p(z)|+\min_{|z|=k}|p(z)|\right]$$
$$\leq(1+k^n)\left[\max_{|z|=1}|p'(z)|-\frac{2|a_1|}{n+1}\left(1-\frac{1}{k^{n-1}}\right)-\frac{2|a_2|}{k^{n-1}}\right.$$
$$\left.\times\left\{\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}\right\}\right],\ \text{for } n>3$$

and

$$3\left[\max_{|z|=1}|p(z)|+\min_{|z|=k}|p(z)|\right]$$
$$\leq(1+k^3)\left[\max_{|z|=1}|p'(z)|-\frac{(k-1)}{2k^2}\{(k+1)|a_1|+(k-1)2|a_2|\}\right],$$
$$\text{for } n=3.$$

The result is best possible and equality in (1.11) *and* (1.12) *holds for* $p(z)=z^n+k^n$.

Remark 3. As mentioned earlier, for $n\geq 3$ Theorem 2 improves upon the result due to Aziz and Ahemad [1], the claim being followed from the same arguments as done in Remark 2.

2. Lemmas

The following lemmas will be needed for the proofs of the theorems.

Lemma 2.1. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* $n \geq 2$, *then for* $R \geq 1$,

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)| - \frac{2(R^n-1)}{n+2}|a_0| - |a_1| \times \left[\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right], \text{ if } n > 2, \tag{2.1}$$

and

$$\max_{|z|=R} |p(z)| \leq R^2 \max_{|z|=1} |p(z)| - \frac{(R-1)}{2} \times [(R+1)|a_0| + (R-1)|a_1|], \text{ if } n = 2\,. \tag{2.2}$$

The above lemma is due to Dewan et al [5].

Lemma 2.2. *If* $p(z)$ *is a polynomial of degree* n *having no zero in* $|z| < 1$, *then for each* $R \geq 1$, *and* $r > 0$,

$$\left\{\int_0^{2\pi} |p(Re^{i\theta})|^r\right\}^{\frac{1}{r}} \leq E_r \left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}, \tag{2.3}$$

where

$$E_r = \frac{\left\{\int_0^{2\pi} |1 + R^n e^{i\alpha}|^r d\alpha\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi} |1 + e^{i\alpha}|^r d\alpha\right\}^{\frac{1}{r}}}.$$

Lemma 2.2 was proved by Boas and Rahman [4] for $r \geq 1$ and then Rahman and Schmeisser [9] have shown that it remains valid for $0 < r < 1$ as well.

3. Proofs of the Theorems

Proof of Theorem 1. Let $n > 3$. Since $p(z)$ has all the zeros in $|z| \leq k$, $k \geq 1$, therefore, the polynomial $G(z) = p(kz)$ has all its zeros in $|z| \leq 1$. Hence the polynomial $H(z) = z^n \overline{G\left(\frac{1}{\bar{z}}\right)}$, has all its zeros in $|z| \geq 1$. Let z_ν, $\nu = 1, 2, \ldots,$

n be the zeros of $H(z)$, then obviously $|z_\nu| \geq 1$, $1 \leq \nu \leq n$ and

$$\frac{zH'(z)}{H(z)} = \sum_{\nu=1}^{n} \frac{z}{z - z_\nu}\,,$$

so that for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are not the zeros of $H(z)$, we have

$$\mathrm{Re}\left(\frac{e^{i\theta}H'(e^{i\theta})}{H(e^{i\theta})}\right) = \sum_{\nu=1}^{n} \mathrm{Re}\left(\frac{e^{i\theta}}{e^{i\theta} - z_\nu}\right) \leq \frac{n}{2}\,.$$

This gives

$$\left|\frac{e^{i\theta}H'(e^{i\theta})}{nH(e^{i\theta})}\right| \leq \left|1 - \frac{e^{i\theta}H'(e^{i\theta})}{nH(e^{i\theta})}\right| \tag{3.1}$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are not the zeros of $H(z)$. Inequality (3.1) is equivalent to

$$|H'(e^{i\theta})| \leq |nH(e^{i\theta}) - e^{i\theta}H'(e^{i\theta})| \tag{3.2}$$

for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are not the zeros of $H(z)$. Since inequality (3.2) is trivially true for points $e^{i\theta}$, $0 \leq \theta < 2\pi$, which are the zeros of $H(z)$ too, therefore, it follows that

$$|H'(z)| \leq |nH(z) - zH'(z)| \quad \text{for} \quad |z| = 1\,. \tag{3.3}$$

Now, since $G(z)$ has all its zeros in $|z| \leq 1$, by Gauss-Lucas Theorem, $G'(z)$ has all its zeros in $|z| \leq 1$ also. This implies that the polynomial

$$z^{n-1}\overline{G'\left(\frac{1}{\bar{z}}\right)} = nH(z) - zH'(z) \tag{3.4}$$

has all its zeros in $|z| \geq 1$, i.e., does not vanish in $|z| < 1$.

Hence from (3.4), it follows that the function

$$w(z) = \frac{zH'(z)}{nH(z) - zH'(z)}$$

is analytic for $|z| \leq 1$ and $|w(z)| \leq 1$ for $|z| \leq 1$. Moreover, $w(0) = 0$ and hence the function $1 + w(z)$ is subordinate to the function $1 + z$ for $|z| \leq 1$. Hence by a well known property of subordination [4], we have for $r > 0$,

$$\int_0^{2\pi} |1 + w(e^{i\theta})|^r d\theta \leq \int_0^{2\pi} |1 + e^{i\theta}|^r d\theta\,. \tag{3.5}$$

Now,

$$1 + w(z) = \frac{nH(z)}{nH(z) - zH'(z)} \tag{3.6}$$

and for $|z| = 1$, we get

$$|G'(z)| = \left| z^{n-1}\overline{G'\left(\frac{1}{\bar{z}}\right)} \right| = |nH(z) - zH'(z)| \quad (\text{by } (3.4)),$$

which implies from (3.6) that

$$\begin{aligned} |nH(z)| &= |1 + w(z)|\,|nH(z) - zH'(z)| \\ &= |1 + w(z)|\,|G'(z)| \quad \text{for} \quad |z| = 1\,. \end{aligned} \tag{3.7}$$

Combining (3.5) and (3.7), we have for $r > 0$,

$$n^r \int_0^{2\pi} |H(e^{i\theta})|^r d\theta \le \int_0^{2\pi} |1 + e^{i\theta}|^r d\theta \left\{ \max_{|z|=1} |G'(z)| \right\}^r . \tag{3.8}$$

Since $H(z)$ has no zero in $|z| \le 1$, therefore, applying Lemma 2.2 to $H(z)$ with $R = k \ge 1$, we obtain for each $r > 0$,

$$\int_0^{2\pi} |H(ke^{i\theta})|^r d\theta \le (B_r)^r \int_0^{2\pi} |H(e^{i\theta})|^r d\theta\,, \tag{3.9}$$

where $B_r = \dfrac{\left\{ \displaystyle\int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}}}{\left\{ \displaystyle\int_0^{2\pi} |1 + e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}}}$.

Since $H(z) = z^n \overline{G\left(\frac{1}{\bar{z}}\right)} = z^n \overline{p\left(\frac{k}{\bar{z}}\right)}$, therefore, for $0 \le \theta < 2\pi$, we have

$$|H(ke^{i\theta})| = |k^n e^{in\theta} \overline{p(e^{i\theta})}| = k^n |p(e^{i\theta})|\,. \tag{3.10}$$

From (3.8), (3.9) and (3.10), it follows for each $r > 0$,

$$\begin{aligned} n^r k^{nr} \int_0^{2\pi} |p(e^{i\theta})|^r d\theta &\le n^r (B_r)^r \int_0^{2\pi} |H(e^{i\theta})|^r d\theta \\ &\le \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \left\{ \max_{|z|=1} |G'(z)| \right\}^r. \end{aligned} \tag{3.11}$$

As the polynomial $G'(z) = kp'(kz)$ is of degree ≥ 3, so using inequality (2.1) of Lemma 2.1, we obtain

$$\begin{aligned} &\max_{|z|=1} |G'(z)| = k \max_{|z|=1} |p'(kz)| = k \max_{|z|=k} |p'(z)| \\ &\le k\left[k^{n-1} \max_{|z|=1} |p'(z)| - \frac{2(k^{n-1} - 1)}{n+1} |a_1| - 2|a_2| \times \left\{ \frac{k^{n-1} - 1}{n-1} - \frac{k^{n-3} - 1}{n-3} \right\} \right] \\ &\le k^n \left[\max_{|z|=1} |p'(z)| - \frac{2\left(1 - \frac{1}{k^{n-1}}\right)}{n+1} |a_1| - 2|a_2| \times \left\{ \frac{1 - \frac{1}{k^{n-1}}}{n-1} - \frac{\left(\frac{1}{k^2} - \frac{1}{k^{n-1}}\right)}{n-3} \right\} \right]. \end{aligned}$$

On combining this with (3.11), we have

$$n^r \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\} \times \left[\max_{|z|=1} |p'(z)| - \frac{2|a_1|}{n+1}\left(1 - \frac{1}{k^{n-1}}\right) - \frac{2|a_2|}{k^{n-1}}\left\{ \frac{(k^{n-1}-1)}{n-1} - \frac{(k^{n-3}-1)}{n-3} \right\} \right]^r .$$

This is equivalent to

$$n\left\{ \int_0^{2\pi} |p(e^{i\theta})|^r d\theta \right\}^{\frac{1}{r}} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}} \times \left[\max_{|z|=1} |p'(z)| - \frac{2|a_1|}{n+1}\left(1 - \frac{1}{k^{n-1}}\right) - \frac{2|a_2|}{k^{n-1}}\left\{ \frac{(k^{n-1}-1)}{n-1} - \frac{(k^{n-3}-1)}{n-3} \right\} \right]$$

which proves inequality (1.5).

The proof of inequality (1.6) follows on the same lines as that of inequality (1.5), but instead of using inequality (2.1) of Lemma 2.1, we use inequality (2.2) of the same lemma. □

Proof of Theorem 2. By hypothesis, all the zeros of $p(z)$ lie in $|z| \leq k$, $k \geq 1$ and $m = \min_{|z|=k} |p(z)|$. If $p(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from Theorem 1. If $p(z)$ has no zero on $|z| = k$, i.e., we suppose that $p(z)$ has all its zeros on $|z| < k$, $k \geq 1$, then $m > 0$.

Let $H(z) = p(kz)$ and $G(z) = z^n \overline{H\left(\frac{1}{\bar{z}}\right)} = z^n \overline{p\left(\frac{k}{\bar{z}}\right)}$, then all the zeros of $G(z)$ lie in $|z| > 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$.

Also $|G(z)| = \left| z^n \overline{p\left(\frac{k}{\bar{z}}\right)} \right| = |p(kz)| \geq m$ for $|z| = 1$.

Hence by the maximum modulus principle, it follows that

$$\left| z^n \overline{p\left(\frac{k}{\bar{z}}\right)} \right| \geq m \quad \text{for} \quad |z| \leq 1 .$$

Replacing z by $\frac{1}{z}$, it follows that

$$|p(kz)| \geq m|z|^n \quad \text{for } |z| \geq 1 ,$$

or

$$|p(z)| \geq m\left|\frac{z}{k}\right|^n \quad \text{for } |z| \geq k . \tag{3.12}$$

Now, consider the polynomial $F(z) = p(z) + \alpha m$, where α is a real or complex number with $|\alpha| \le 1$, then all the zeros of $F(z)$ lie in $|z| \le k$. Because if for some $z = z_0$ with $|z_0| > k$, we have $F(z_0) = p(z_0) + \alpha m = 0$, then

$$|p(z_0)| = |\alpha m| \le m < m \left|\frac{z_0}{k}\right|^n$$

which is a contradiction to (3.12). Hence for every real or complex number α with $|\alpha| \le 1$, the polynomial $F(z) = p(z) + \alpha m$ has its zeros in $|z| \le k$, where $k \ge 1$. Applying Theorem 1 to the polynomial $F(z)$ and noting that $F'(z) = p'(z)$, Theorem 2 follows. □

References

[1] A. Aziz, N. Ahemad, Integral mean estimates for polynomials whose zeros are within a circle, *Glasnik Matematički*, **51** (1996), 229-237.

[2] A. Aziz, Inequalities for the derivative of a polynomial, *Proc. Amer. Math. Soc.*, **89** (1983), 259-266.

[3] A. Aziz, Integral mean estimates for polynomials with restricted zeros, *J. Approx. Theory*, **55** (1988), 232-239.

[4] R.P. Boas Jr., Q.I. Rahman, L^p inequalities for polynomials and entire functions, *Arch. Rational Mech. Anal.*, **11** (1962), 34-39.

[5] K.K. Dewan, J. Kaur, A. Mir, Inequalities for the derivative of a polynomial, *J. Math. Anal. and Appl.*, **269** (2002), 489-499.

[6] N.K. Govil, On the derivative of a polynomial, *Proc. Amer. Math. Soc.*, **41** (1973), 543-546.

[7] N.K. Govil, Some inequalities for derivatives of polynomials, *J. Approx. Theory*, **66** (1991), 29-35.

[8] M.A. Malik, An integral mean estimate for polynomials, *Proc. Amer. Math. Soc.*, **91** (1984), 281-284.

[9] Q.I. Rahman, G. Schmeisser, L^p inequalities for polynomials, *J. Approx. Theory*, **53** (1988), 26-32.

[10] P. Turán, Über die Ableitung von Polynomen, *Compositio Math.*, **7** (1939), 89-95.