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ON THE ENESTRÖM-KAKEYA THEOREM

N.A. Rather[1 §], Shakeel A. Simnani[2], M.I. Mir[3]

[1]Department of Mathematics
University of Kashmir
Hazratbal, Srinagar, 190006, INDIA
e-mail: nisararather@yahoo.co.in

[2]Department of Physics
University of Kashmir
Hazratbal, Srinagar, 190006, INDIA
e-mail: shakeel@kashmiruniversity.net

[3]Department of Mathematics
National Institute of Technology (NIT)
Hazratbal, Srinagar 190006, INDIA

Abstract: In this paper, we present some generalization of a well-known result of Eneström and Kakeya concerning the bounds for the moduli of the zeros of polynomials with complex coefficients.

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1. Introduction and Statement of Results

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0,$$

then according to a well-known result of Eneström and Kakeya (see [13], [14]), the polynomial $P(z)$ has all its zeros in $|z| \leq 1$.

We may apply this result to the polynomial $P(tz)$ to obtain the following more general result.

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§Correspondence author

Theorem A. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients such that for some $t > 0$,*

$$t^n a_n \geq t^{n-1} a_{n-1} \geq \cdots \geq t_1 a_1 \geq a_0 > 0,$$

then all the zeros of $P(z)$ lie in $|z| \leq t$.

In the literature [1], [2], [5]-[14], there exists several extensions and generalization of the Eneström-Kakeya Theorem. A. Aziz and Q.G. Mohammad [3] used Schwarz Lemma to obtain the following generalization of Theorem A.

Theorem B. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real and positive coefficients. If $t_1 > t_2 \geq 0$ can be found such that*

$$a_r t_1 t_2 + a_{r-1}(t_1 - t_2) - a_{r-2} \geq 0, \quad r = 1, 2, \ldots, n+1,$$

$a_{-1} = a_{n+1} = 0$, then all the zeros of $P(z)$ lie in $|z| \leq t_1$.

For $t_2 = 0$, this reduces to Theorem A.

N.K. Govil and Q.I. Rahman [9] extended the Eneström-Kakeya to the polynomials with complex coefficients by proving:

Theorem C. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that*

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2, \quad j = 0, 1, .., n,$$

for some real β and

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ lie in

$$|z| \leq (\text{Cos}\alpha + \text{Sin}\alpha) + \frac{2\text{Sin}\alpha}{|a_n|} \sum_{j=0}^{n} |a_j|.$$

In this paper we first present the following result which is an interesting extension of Theorem B to the polynomials having complex coefficients.

Theorem 1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $t_1 > t_2 \geq 0$ can be found such that*

$$t_1 t_2 |a_r| + (t_1 - t_2)|a_{r-1}| - |a_{r-2}| \geq 0, \quad r = 1, 2, .., k+1,$$

and

$$t_1 t_2 |a_r| + (t_1 - t_2)|a_{r-1}| - |a_{r-2}| \leq 0, \quad r = k+2, .., n+1,$$

$0 \leq k \leq n$, $a_{-1} = a_{n+1} = 0$, then all the zeros of $P(z)$ lie in

$$|z| \leq t_1 \left\{ \frac{2|a_k| + 2t_2|a_{k+1}|}{t_1^{n-k}|a_n|} - 1 \right\} + \frac{2}{|a_n|} \sum_{j=0}^{n} \frac{|a_j - |a_j||}{t^{n-j-1}}.$$

The following corollary, which ia a generalization of Theorem B, follows by taking $k = n$ in Theorem 1.

Corollary 1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $t_1 > t_2 \geq 0$ can be found such that*

$$t_1 t_2 |a_r| + (t_1 - t_2)|a_{r-1}| - |a_{r-2}| \geq 0, \quad r = 1, 2, .., n+1,$$

$a_{-1} = a_{n+1} = 0$, then all the zeros of $P(z)$ be in

$$|z| \leq t_1 + \frac{2}{|a_n|} \sum_{j=0}^{n} \frac{|a_j - |a_j||}{t_1^{n-j-1}}.$$

For real and positive $a_j, j = 0, 1, .., n$, Corollary 1 reduces to Theorem B. Also for $t_2 = 0$, Theorem 1 reduces to a result due to Aziz and Mohammad [4, Theorem 4].

It is interesting to eximine the bound if in addition to the hypothesis of Theorem 1, the coefficients a_j of $P(z)$ are such that

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2, \quad j = 0, 1, 2, .., n,$$

for some real β. In this direction, we next prove:

Theorem 2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that*

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2, \quad j = 0, 1, 2, .., n,$$

for some real β. If $t_1 > t_2 \geq 0$ can be found that

$$t_1 t_2 |a_r| + (t_1 - t_2)|a_{r-1}| - |a_{r-2}| \geq 0, \quad r = 1, 2, .., k+1$$

and

$$t_1 t_2 |a_r| + (t_1 - t_2)|a_{r-1}| - |a_{r-2}| \leq 0, \quad r = k+2, .., n+1,$$

$0 \leq k \leq n$, $a_{-1} = a_{n+1} = 0$, then all the zeros of $P(z)$ lie in

$$|z| \leq t_1 \left\{ \left(\frac{2|a_k| + 2t_2|a_{k+1}|}{t_1^{n-k}|a_n|} - 1 \right) \text{Cos}\alpha + \text{Sin}\alpha \right\} + 2\frac{\text{Sin}\alpha}{|a_n|} \sum_{j=0}^{n-1} \frac{|a_j|}{t_1^{n-j-1}}.$$

For $k = n$, we get:

Corollary 2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that*

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2, \quad j = 0, 1, 2, .., n,$$

for some real β. If $t_1 > t_2 \geq 0$ can be found such that

$$t_1 t_2 |a_r| + (t_1 - t_2)|a_{r-1}| - |a_{r-2}| \geq 0, \quad r = 1, 2, .., n+1,$$

$a_{-1} = a_{n+1} = 0$, *then all the zeros of* $P(z)$ *lie in*

$$|z| \leq t_1(\text{Cos}\alpha + \text{Sin}\alpha) + \frac{\text{Sin}\alpha}{|a_n|}\sum_{j=0}^{n-1}\frac{|a_j|}{t_1^{n-j-i}}.$$

Theorem B and Theorem C are both special cases of Corollary 2. Finally we state the following two generalizations of Theorem B. As their proofs are almost similar to the proof of Theorem 1, we omit the details.

Theorem 3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $\text{Re}a_j = \alpha_j$ *and* $\text{Im}a_j = \beta_j$, $j = 0, 1, \ldots, n$. *If* $t_1 > t_2 \geq 0$ *can be found such that*

$$t_1 t_2 \alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \geq 0, \quad r = 1, 2, \ldots, k+1,$$

and

$$t_1 t_2 \alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \leq 0, \quad r = k+2, .., n+1,$$

$0 \leq k \leq n$, $\alpha_{-1} = \alpha_{n+1} = 0$, $\alpha_n > 0$, *then all the zeros of* $P(z)$ *lie in*

$$|z| \leq t_1\left\{\frac{2\alpha_k + t_2\alpha_{k+1}}{t_1^{n-k}\alpha_n} - 1\right\} + \frac{2}{\alpha_n}\sum_{j=0}^{n}\frac{|\beta_j|}{t_1^{n-j-1}}.$$

Theorem 4. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $\text{Re}\alpha_j = \alpha_j$ *and* $\text{Im}a_j = \beta_j$, $j = 0, 1, \ldots, n$. *If* $t_1 > t_2 \geq 0$ *can be found such that*

$$t_1 t_2 \alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \geq 0, \quad r = 1, 2, \ldots, k+1,$$

$$t_1 t_2 \alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \leq 0, \quad r = k+2, .., n+1, \quad 0 \leq k \leq n,$$

and

$$t_1 t_2 \beta_r + (t_1 - t_2)\beta_{r-1} - \beta_{r-2} \geq 0, \quad r = 1, 2, \ldots, n+1,$$

$$t_1 t_2 \beta_r + (t_1 - t_2)\beta_{r-1} - \beta_{r-2} \leq 0, \quad r = m+2, \ldots, m+1, \quad 0 \leq m \leq n,$$

$\alpha_1 = \beta_1 = \alpha_{n+1} = \beta_{n+1} = 0$, $\alpha_n, \beta_n > 0$, *then all the zeros of* $P(z)$ *lie in*

$$|z| \leq \frac{t}{|a_n|}\left\{2t_1^{k-n}(\alpha_k + t_2\alpha_{k+1}) + 2t_1^{m-n}(\beta_m + t_2\beta_{m+1}) - (\alpha_n + \beta_n)\right\}.$$

If we set $k = m = n$ in Theorem 4, we obtain:

Corollary 3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with* $\text{Re}\alpha_j = \alpha_j$ *and* $\text{Im}a_j = \beta_j$, $, j = 0, 1, \ldots, n$. *If* $t_1 > t_2 \geq 0$ *can be found such that*

$$t_1 t_2 \alpha_r + (t_1 - t_2)\alpha_{r-1} - \alpha_{r-2} \geq 0, \quad r = 1, 2, .., n+1$$

and

$$t_1t_2\beta_r + (t_1 - t_2)\beta_{r-1} - \beta_{r-2} \geq 0, \ \ r = 1, 2, \ldots, n+1,$$

$\alpha_1 = \beta_1 = \alpha_{n+1} = \beta_{n+1} = 0$, $\alpha_n > 0, \beta_n > 0$, *then all the zeros of* $P(z)$ *lie in*

$$|z| \leq t_1\left\{\frac{\alpha_n + \beta_n}{|a_n|}\right\} \leq \sqrt{2}t_1.$$

2. Lemmas

For the proofs of these theorem, we need following lemmas.

Lemma 1. (see [4]) *Let* $P(z) = a_nz^n + a_pz^p + \cdots + a_1z + a_0, 0 \leq p \leq n-1$, *be a polynomial of degree n with complex coefficients. Then for every positive real number r, all the zeros of* $P(z)$ *lie in the circle*

$$|z| \leq \text{Max}\left\{r, \sum_{j=0}^{p} |\frac{a_j}{a_n}| \frac{1}{r^{n-j-1}}\right\}.$$

Lemma 2. *If* $|\text{arg } a_j - \beta| \leq \alpha \leq \pi/2, \ \ j = 0, 1, 2, .., n$, *and* β *real, then for* $t_1 > t_2 \geq 0$,

$$|t_1t_2a_j + (t_1 - t_2)a_{j-1} - a_{j-2}| \leq |t_1t_2a_j| + (t_1 - t_2)|a_{j-1}| - |a_{j-2}||\text{Cos}\alpha + (t_1t_2|a_j| + (t_1 - t_2)|a_{j-1}| + |a_{j-2}|)\text{Sin}\alpha.$$

Proof of Lemma 2. Let arg $a_j = \alpha_j$, arg $a_{j-1} = \alpha_{j-1}$ and arg $a_{j-2} = \alpha_{j-2}$, then

$$|t_1t_2a_j + (t_1 - t_2)a_{j-1} - a_{j-2}|^2$$
$$= |t_1t_2|a_j|e^{i\alpha_j} + (t_1 - t_2)|a_{j-1}|e^{i\alpha_{j-1}} - |a_{j-2}|e^{i\alpha_{j-2}}|^2$$
$$= t_1^2t_2^2|a_j|^2 + (t_1 - t_2)^2|a_{j-1}|^2 + |a_{j-2}|^2 + 2t_1t_2(t_1 - t_2)|a_j||a_{j-1}|\text{Cos}(\alpha_j - \alpha_{j-1}) - 2(t_1 - t_2)|a_{j-1}||a_{j-2}|\text{Cos}(\alpha_{j-1} - \alpha_{j-2}) - 2t_1t_2|a_{j-2}||a_j|\text{Cos}(\alpha_{j-2} - \alpha_j).$$

Since by hypothesis, $|\alpha_j - \beta| \leq \alpha \leq \pi/2$, $|\alpha_{j-1} - \beta| \leq \alpha \leq \pi/2$ and $|\alpha_{j-2} - \beta| \leq \alpha \leq \pi/2$, therefore,

$$|\alpha_{j-1} - \alpha_{j-2}| = |\alpha_{j-1} - \beta + \beta - \alpha_{j-2}| \leq |\alpha_{j-1} - \beta| + |\alpha_{j-2} - \beta| \leq 2\alpha \leq \pi.$$

Hence

$$\text{Cos}(\alpha_{j-1} - \alpha_{j-2}) \geq \text{Cos}2\alpha. \quad (1)$$

Similarly, we can prove

$$\mathrm{Cos}(\alpha_{j-2} - \alpha_j) \geq \mathrm{Cos}2\alpha. \tag{2}$$

Using (2) and (3) in (1), we get

$$\begin{aligned}
|t_1t_2a_j + (t_1 - t_2)a_{j-1} - a_{j-2}|^2 \leq\ & t_1^2t_2^2|a_j|^2 + (t_1 - t_2)^2|a_{j-1}|^2 + |a_{j-2}|^2 \\
& + 2t_1t_2(t_1 - t_2)|a_j||a_{j-1} - 2(t_1 - t_2)|a_{j-1}||a_{j-2}|\mathrm{Cos}2\alpha \\
& - 2(t_1 - t_2)|a_{j-1}||a_{j-2}|\mathrm{Cos}2\alpha - 2t_1t_2|a_{j-2}||a_j|\mathrm{Cos}2\alpha \\
=\ & |t_1t_2|a_j| + (t_1 - t_2)|a_{j-1}| - |a_{j-2}||^2\mathrm{Cos}^2\alpha \\
& + (t_1t_2|a_j| + (t_1 - t_2)|a_{j-1}| + |a_{j-2}|)^2\mathrm{Sin}^2\alpha \\
\leq\ & \Big\{ |t_1t_2|a_j| + (t_1 - t_2)|a_{j-1}| - |a_{j-2}||\mathrm{Cos}\alpha \\
& + (t_1t_2|a_j| + (t_1 - t_2)|a_{j-1}| + |a_{j-2}||)\mathrm{Sin}\alpha \Big\}^2,
\end{aligned}$$

since

$$2\mathrm{Sin}\alpha\mathrm{Cos}\alpha = \mathrm{Sin}2\alpha \geq 0\,, \quad \text{for} \quad 0 \leq 2\alpha \leq \pi.$$

This gives,

$$\begin{aligned}
|t_1t_2a_j + (t_1 - t_2)a_{j-1} - a_{j-2}| \leq\ & |t_1t_2|a_j| + (t_1 - t_2)|a_{j-1}| - |a_{j-2}||\mathrm{Cos}\alpha \\
& + (t_1t_2|a_j| + (t_1 - t_2)|a_{j-1}| + |a_{j-2}|)\mathrm{Sin}\alpha,
\end{aligned}$$

which proves Lemma 2. □

Proof of Theorem 1. Consider the polynomial

$$\begin{aligned}
F(z) &= (t_2 + z)(t_1 - z)P(z) = (t_2 + z)(t_1 - z)(a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0) \\
&= -a_nz^{n+2} + (a_nt_1t_2 + (t_1 - t_2)a_{n-1})z^{n+1} + (a_{n-1}t_1t_2 + (t_1 - t_2)a_{n-2} - a_{n-3})z^n \\
&\quad + \cdots + (a_2t_1t_2 + (t_1 - t_2)a_1 - a_0)z^2 + (a_1t_1t_2 + (t_1 - t_2)a_0) + a_0t_1t_2 \\
&= -a_nz^{n+2} + \sum_{j=0}^{n+1}(t_1t_2a_j + (t_1 - t_2)a_{j-1} - a_{j-2})z^j \quad (a_{-1} = a_{-2} = a_{n+1} = 0).
\end{aligned}$$

Applying Lemma 1 to the polynomial $F(z)$ which is of degree $n+2$ with $r = t_1$ and $p = n + 1$, it follows that all the zeros of $F(z)$ lie in

$$\begin{aligned}
|z| &\leq Max\left(t_1, \sum_{j=0}^{n+1} \frac{|t_1t_2a_j + (t_1 - t_2)a_{j-1} - a_{j-2}|}{t_1^{n-j+1}|a_n|}\right) \\
&= \sum_{j=0}^{n+1} \frac{|t_1t_2a_j + (t_1 - t_2)a_{j-1} - a_{j-2}|}{t_1^{n-j+1}|a_n|},
\end{aligned}$$

since

$$t_1 = \left| \sum_{j=0}^{n+1} \frac{t_1 t_2 a_j + (t_1 - t_2) a_{j-1} - a_{j-2}}{t_1^{n-j+1} a_n} \right| \leq \sum_{j=0}^{n+1} \frac{|t_1 t_2 a_j + (t_1 - t_2) a_{j-1} - a_{j-2}|}{t_1^{n-j+1} |a_n|}.$$

Now

$$\begin{aligned}
&\sum_{j=0}^{n+1} \frac{|t_1 t_2 a_j + (t_1 - t_2) a_{j-1} - a_{j-2}|}{t_1^{n-j+1} |a_n|} \\
&\quad \leq \sum_{j=0}^{n+1} \frac{|t_1 t_2 |a_j| + (t_1 - t_2)|a_{j-1}| - |a_{j-2}||}{t_1^{n-j+1} |a_n|} \\
&\quad + \sum_{j=0}^{n+1} \frac{t_1 t_2 |a_j - |a_j|| + (t_1 - t_2)|a_{j-1} - |a_{j-1}|| + |a_{j-2} - |a_{j-2}||}{t_1^{n-j+1} |a_n|} \\
&\quad = \sum_{j=0}^{k+1} \frac{(t_1 t_2 |a_j| + (t_1 - t_2)|a_{j-1}| - |a_{j-2}|)}{t_1^{n-j+1} |a_n|} \\
&\quad + \sum_{j=k+2}^{n+1} \frac{(|a_{j-2}| - (t_1 - t_2)|a_{j-1}| - t_1 t_2 |a_j|)}{t_1^{n-j+1} |a_n|} \\
&\quad + \sum_{j=0}^{n+1} \frac{t_1 |a_{j-1} - |a_{j-1}|| + |a_{j-2} - |a_{j-2}||}{t_1^{n-j+1} |a_n|} \\
&= \left(\frac{2|a_k|}{t_1^{n-k+1} |a_n|} + \frac{2t_2 |a_{k+1}|}{t_1^{n-k+1} |a_n|} - t_1 \right) + \sum_{j=0}^{n+1} \frac{t_1 |a_{j-1} - |a_{j-1}|| + |a_{j-2} - |a_{j-2}||}{t_1^{n-j+1} |a_n|} \\
&\quad \leq t_1 \left\{ \frac{2|a_k| + 2t_2 |a_{k+1}|}{t_1^{n-k} |a_n|} - 1 \right\} + 2 \sum_{j=0}^{n} \frac{|a_j - |a_j||}{t^{n-j-1} |a_n|}.
\end{aligned}$$

Therefore, it follows from (4) that all zeros of $F(z)$ lie in

$$|z| \leq t_1 \left\{ \frac{2|a_k| + 2t_2 |a_{k+1}|}{t_1^{n-k} |a_n|} - 1 \right\} + 2 \sum_{j=0}^{n} \frac{|a_j - |a_j||}{t^{n-j-1} |a_n|}.$$

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, the theorem is proved. □

Proof of Theorem 2. Proceedings similarly as in the proof of Theorem 1, it follows from (4) by using Lemma 2, that all the zeros of $F(z)$ lie in

$$|z| \leq \sum_{j=0}^{n+1} \frac{|t_1t_2|a_j| + (t_1-t_2)|a_{j-1}| - |a_{j-2}||\text{Cos}\alpha}{t_1^{n-j+1}|a_n|} + \sum_{j=0}^{n+1} \frac{(t_1t_2|a_j| + (t_1-t_2)|a_{j-1}| + |a_{j-2})\text{Sin}\alpha}{t_1^{n-j+1}|a_n|}.$$

Now,

$$\begin{aligned}
\sum_{j=0}^{n+1} & \frac{|t_1t_2|a_j| + (t_1-t_2)|a_{j-1}| - |a_{j-2}||\text{Cos}\alpha}{t_1^{n-j+1}|a_n|} \\
&= \sum_{j=0}^{k+1} \frac{\left(t_1t_2|a_j| + (t_1-t_2)|a_{j-1}| - |a_{j-2}\right)}{t_1^{n-j+1}|a_n|}\text{Cos}\alpha \\
&\quad - \sum_{j=k+2}^{n+1} \frac{\left(t_1t_2|a_j| + (t_1-t_2)|a_{j-1}| - |a_{j-2}\right)}{t_1^{n-j+1}|a_n|}\text{Cos}\alpha \\
&= t_1\left(\frac{|a_k| + t_2|a_{k+1}|}{t_1^{n-k}|a_n|}\right)\text{Cos}\alpha - t_1\left(\frac{-|a_k| - t_2|a_{k+1}|}{t_1^{n-k}|a_n|} + 1\right)\text{Cos}\alpha \\
&= t_1\left(\frac{2|a_k| + 2t_2|a_{k+1}|}{t_1^{n-k}|a_n|}\right)\text{Cos}\alpha - t_1\text{Cos}\alpha.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{j=0}^{n+1} \frac{\left(t_1t_2|a_j| + (t_1-t_2)|a_{j-1}| + |a_{j-2}|\right)}{t_1^{n-j+1}|a_n|}\text{Sin}\alpha &= \sum_{j=0}^{n+1} \frac{t_1|a_{j-1}| + |a_{j-2}|}{t_1^{n-j+1}|a_n|}\text{Sin}\alpha \\
&= 2\sum_{j=0}^{n-1} \frac{|a_j|}{t_1^{n-j-1}|a_n|}\text{Sin}\alpha + t_1\text{Sin}\alpha.
\end{aligned}$$

Therefore from (5), we conclude that all the zeros of $F(z)$ and hence that of $P(z)$ lie in

$$|z| \leq t_1\left\{\left(\frac{2|a_k| + 2t_2|a_{k+1}|}{t_1^{n-k}|a_n|} - 1\right)\text{Cos}\alpha + \text{Sin}\alpha\right\} + 2\frac{\text{Sin}\alpha}{|a_n|}\sum_{j=0}^{n-1} \frac{|a_j|}{t_1^{n-j-1}}.$$

This completes the proof of Theorem 2. □

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