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ON THE NUMBER OF DOMINATING SETS IN SOME CLASSES OF TREES

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Abstract: A dominating set of a graph G is a vertex subset S such that every vertex of G either is in S or is adjacent to a vertex in S. The number of dominating sets in special kinds of trees is determined. Some of these results are given by recurrence relations.

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1. Introduction

For general concepts we refer the reader to Diestel [2]. Only simple graphs G are considered. $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. P_n denotes a *path* on n vertices. The *complete graph* on n vertices is denoted by K_n, while $\overline{K_n}$ stands for the complement of K_n, i.e. the graph consisting of n isolated vertices. A *tree* T is a connected graph with no cycles. A *leaf* of T is a vertex of degree one in T. By a *branch* B at a vertex x in a tree T we mean a maximal subtree which includes x and exactly one edge incident to x. Let $d_i \geq 2$, $i = 1, 2$, be integers. Let K_{1,d_1-1} and K_{1,d_2-1} be stars with central vertices y_1 and y_2, respectively. By *double palm* we mean a graph P_{n,d_1,d_2} with $n \geq d_1+d_2$ defined as follows: for $n = d_1+d_2$ we have $V(P_{n,d_1,d_2}) = V(K_{1,d_1-1}) \cup V(K_{1,d_2-1})$ and $E(P_{n,d_1,d_2}) = E(K_{1,d_1-1}) \cup E(K_{1,d_2-1}) \cup \{y_1y_2\}$, for $n > d_1+d_2$ we have $V(P_{n,d_1,d_2}) = V(K_{1,d_1-1}) \cup V(K_{1,d_2-1}) \cup V(P_{n-(d_1+d_2)})$ and $E(P_{n,d_1,d_2}) = E(K_{1,d_1-1}) \cup E(K_{1,d_2-1}) \cup E(P_{n-(d_1+d_2)}) \cup \{x_1y_1\} \cup \{x_{n-(d_1+d_2)}y_2\}$.

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If $d_1 = 2$ and $d_2 > 2$ then the double palm we call *palm* and denote it by P_{n,d_2}. By the *root of a star* or *of a path* we mean any of its leaves. By the *root of a palm* $P_{n,d}$ with $2 < d < n-1$ we mean the only leaf whose neighbor is of degree 2.

The symbol nG denotes union of n, $n \geq 2$, disjoint copies of the graph G. Let G, H be graphs with $|V(G)| = n$. The *corona* $G \circ H$ of G and H is the graph obtained from the disjoint union of G and nH by joining the i th vertex of the copy of G to every vertex in the i th copy of H. Let $U \subset V(G)$. Then $G - U$ is a graph obtained from G by deleting all the vertices in $U \cap V(G)$ and their incident edges. If $U = \{v\}$ is a singleton, we will write $G - v$.

In a graph G, an *independent set* is a subset Q of $V(G)$ such that no two vertices in Q are adjacent. A subset $S \subseteq V(G)$ is called a *dominating set* of G if any vertex of G either is in S or is adjacent to a vertex in S. Certainly, every graph has a dominating set. Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in Haynes et al [3, 4]. Let $\partial(G)$ stand for the number of dominating sets in G. For a given set $X \subset V(G)$, let $\partial_X(G)$ denote the number of all dominating sets in G with X included in each of the dominating sets. If $X = \{x\}$ then we will write $\partial_x(G)$. On the other hand, let $\partial_{\hat{x}}(G)$ be the number of dominating sets in G which all avoid the vertex x. Then the basic rule for counting dominating sets in a graph G is as follows

$$\partial(G) =_x \partial_x(G) + \partial_{\hat{x}}(G) \qquad \text{for any } x \in V(G). \tag{1}$$

The symbol $=_x$ denotes that the right hand side of the formula (1) presents the sum of two summands ∂_x and $\partial_{\hat{x}}$ in this order.

Theorem 1. (see [1]) *Let T be a tree with vertex x of degree b, $b > 1$, and with b branches B_i all at x. Assume that $L(x)$ is the set of leaves attached to x, $|L(x)| = \ell$, $1 \leq \ell \leq b$. Then*

$$\partial(T) =_x 2^{\ell} \partial_x(T - L(x)) + \partial(T - L(x) - x)$$

$$=_x 2^{\ell} \prod_{i=1}^{b-\ell} \partial_x(B_i) + \prod_{i=1}^{b-\ell} \partial(B_i - x).$$

Theorem 2. (see [1]) *Let $p_n = \partial(P_n)$, $p_n^* = \partial_{z_1}(P_n)$, $p_n^{**} = \partial_{\{z_1,z_2\}}(P_n)$ where z_1, z_2 are leaves of P_n. Then:*

a) $p_n = p_{n-1} + p_{n-2} + p_{n-3}$ with initial conditions $p_0 = p_1 = 1$, $p_2 = 3$,

b) $p_n^* = p_{n-1}^* + p_{n-2}^* + p_{n-3}^*$ with initial conditions $p_i^* = i$ for $i = 0, 1, 2$,

c) $p_n^{**} = p_{n-1}^{**} + p_{n-2}^{**} + p_{n-3}^{**}$ with initial conditions $p_0^{**} = 0$, $p_1^{**} = p_2^{**} = 1$.

Proposition 1. (see [1]) *Let $n \geq 5$, $d \geq 1$ be integers. Then* $\partial(P_{n,d}) = 2^{d-1}p^*_{n-d+1} + p_{n-d}$.

Proposition 2. (see [1]) *Let $n \geq 5$, $2 < d < n-1$ be integers. Assume that r is the root of $P_{n,d}$. Then* $\partial_r(P_{n,d}) = 2^{d-1}p^{**}_{n-d+1} + p^*_{n-d}$.

2. Main Results

Lemma 1. *Let $p^{**}_n = \partial_{\{z_1,z_2\}}(P_n)$ where z_1, z_2 are leaves of P_n. Then for $n \geq 2$, $p^{**}_n = \frac{1}{2}(p^*_{n+1} - p^*_{n-1})$ with initial conditions $p^*_i = i$ for $i = 1, 2, 3$.*

Proof. Initial conditions are obvious. Let $n \geq 4$. It is easily seen that $p^{**}_n = p^{**}_{n-1} + p^*_{n-2}$. By Theorem 2, we have $p^{**}_{n-1} + p^{**}_{n-2} + p^{**}_{n-3} = p^{**}_{n-1} + p^*_{n-2}$. Hence, $p^{**}_{n-3} + p^*_{n-4} + p^{**}_{n-3} = p^*_{n-2}$. Consequently, $p^{**}_{n-3} = \frac{1}{2}(p^*_{n-2} - p^*_{n-4})$, which ends the proof. □

Theorem 3. *Let $n \geq 6$, $d_i \geq 3$, $i = 1, 2$, be integers. Then* $\partial(P_{n,d_1,d_2}) = 2^{d_1+d_2-3}p^*_{n-d_1-d_2+3} + (2^{d_1-1} + 2^{d_2-1} - 2^{d_1+d_2-3})p^*_{n-d_1-d_2+1} + p_{n-d_1-d_2}$.

Proof. Let n, d_1, d_2 be as in the statement of the theorem. Assume that x is the vertex of degree d_1 of the double palm P_{n,d_1,d_2}. By Theorem 1, $\partial(P_{n,d_1,d_2}) =_x 2^{d_1-1}\partial_x(P_{n-d_1+1,d_2}) + \partial(P_{n-d_1,d_2})$. Using Propositions 1 and 2 gives $\partial(P_{n,d_1,d_2}) = 2^{d_1-1}(2^{d_2-1}p^{**}_{n-d_1-d_2+2} + p^*_{n-d_1-d_2+1}) + 2^{d_2-1}p^*_{n-d_1-d_2+1} + p_{n-d_1-d_2} = 2^{d_1+d_2-2}p^{**}_{n-d_1-d_2+2} + (2^{d_1-1} + 2^{d_2-1})p^*_{n-d_1-d_2+1} + p_{n-d_1-d_2}$
$= 2^{d_1+d_2-3}(p^*_{n-d_1-d_2+3} - p^*_{n-d_1-d_2+1}) + (2^{d_1-1} + 2^{d_2-1})p^*_{n-d_1-d_2+1} + p_{n-d_1-d_2} = 2^{d_1+d_2-3}p^*_{n-d_1-d_2+3} + (2^{d_1-1} + 2^{d_2-1} - 2^{d_1+d_2-3})p^*_{n-d_1-d_2+1} + p_{n-d_1-d_2}$, which ends the proof. □

Corollary 1. *Let $n \geq 6$, $d \geq 3$ be integers. Then* $\partial(P_{n,d,n-d}) = (1 + 2^{d-1})(1 + 2^{n-d-1})$. □

Theorem 4. *Let $n \geq 6$, $d_1 \geq d_2 \geq 3$ be integers. Then* $\partial(P_{n,d_1,d_2}) < \partial(P_{n,d_1+1,d_2-1})$.

Proof. By Th. 3, $\partial(P_{n,d_1+1,d_2-1}) - \partial(P_{n,d_1,d_2}) = 2^{d_1+d_2-3}p^*_{n-d_1-d_2+3} + (2^{d_1} + 2^{d_2-2} - 2^{d_1+d_2-3})p^*_{n-d_1-d_2+1} + p_{n-d_1-d_2} - 2^{d_1+d_2-3}p^*_{n-d_1-d_2+3} - (2^{d_1-1} + 2^{d_2-1} - 2^{d_1+d_2-3})p^*_{n-d_1-d_2+1} - p_{n-d_1-d_2} = (2^{d_1-1} - 2^{d_2-2})p^*_{n-d_1-d_2+1} > 0$. □

Lemma 2. (see [1]) *If z is one of two leaves with a common neighbor in a tree T, then $\partial_z(T) = \partial(T - z)$.*

Theorem 5. *Let $n \geq 6$, $d_i \geq 3$, $i = 1, 2$, be integers and $d_1 > d_2$. Let*

x, y be vertices of degree d_1, d_2 in P_{n,d_1,d_2}, respectively. Assume that z_1, z_2 are leaves which are adjacent to vertices x, y, respectively. Then $\partial_{z_2}(P_{n,d_1,d_2}) > \partial_{z_1}(P_{n,d_1,d_2})$.

Proof. Let $d_1 = deg\, x$, $d_2 = deg\, y$. Assume that z_1, z_2 are leaves at x, y, respectively. Then, by Lemma 2, $\partial_{z_1}(P_{n,d_1,d_2}) = \partial(P_{n,d_1,d_2} - z_1) = \partial(P_{n-1,d_1-1,d_2})$. Using Th. 3 gives $\partial(P_{n-1,d_1-1,d_2}) = 2^{d_1+d_2-4}p^*_{n-d_1-d_2+3} + (2^{d_1-2} + 2^{d_2-1} - 2^{d_1+d_2-4})p^*_{n-d_1-d_2+1} = p_{n-d_1-d_2}$.

Similarly for z_2 we obtain $\partial_{z_2}(P_{n,d_1,d_2}) = \partial(P_{n-1,d_1,d_2-1}) = 2^{d_1+d_2-4}\, p^*_{n-d_1-d_2+3} + (2^{d_1-1} + 2^{d_2-2} - 2^{d_1+d_2-4})p^*_{n-d_1-d_2+1} = p_{n-d_1-d_2}$.

Hence $\partial_{z_2}(P_{n,d_1,d_2}) - \partial_{z_1}(P_{n,d_1,d_2}) = (2^{d_1-1} + 2^{d_2-2} - 2^{d_1+d_2-4} - 2^{d_1-2} - 2^{d_2-1} + 2^{d_1+d_2-4})p^*_{n-d_1-d_2+1} = (2^{d_1-2} - 2^{d_2-2})p^*_{n-d_1-d_2+1} = \frac{1}{4}(2^{d_1} - 2^{d_2})\, p^*_{n-d_1-d_2+1} > 0$, which completes the proof. □

Now we calculate the number $\partial(P_n \circ \overline{K_p})$, $p \geq 0$. Denote $G^p_n = P_n \circ \overline{K_p}$. In Startek et al [5] it was determined the number of independent sets of G^p_n.

Theorem 6. (see [5]) *Let $n \geq 0$, $p \geq 1$ be integers. Let $F(G^p_n)$ denote the number of independent sets of $G^p_n = P_n \circ \overline{K_p}$. Then $F(G^p_n) = 2^p F(G^p_{n-1}) + 2^p F(G^p_{n-2})$ with initial conditions $F(G^p_0) = 1$ and $F(G^p_1) = 2^p + 1$.*

Theorem 7. *Let $n \geq 0$, $p \geq 1$ be integers. Denote $G^p_n = P_n \circ \overline{K_p}$. Then $\partial(G^p_n) = (2^p + 1)\partial(G^p_{n-1})$ with initial condition $\partial(G^p_0) = 1$.*

Proof. If $n = 0$ then $G^p_0 = P_0 \circ \overline{K_p}$ is the empty graph and the empty set is the unique dominating set of this graph. Assume now that $n \geq 1$. Let the set $V(P_n) = \{x_1, x_2, \ldots, x_n\}$, $n \geq 1$, be numbered in the natural fashion. Assume that $L(x_1) = \{y^1_1, y^2_1, \ldots, y^p_1\}$ is the set of leaves attached to x_1. Then, by Theorem 1, $\partial(G^p_n) =_{x_1} 2^p \partial_{x_1}(G^p_n - \{y^1_1, y^2_1, \ldots, y^p_1\}) + \partial(G^p_n - \{y^1_1, y^2_1, \ldots, y^p_1, x_1\})$. Since x_1 is a leaf of $G^p_n - \{y^1_1, y^2_1, \ldots, y^p_1\}$, by Lemma 2 we obtain $\partial_{x_1}(G^p_n - \{y^1_1, y^2_1, \ldots, y^p_1\}) = \partial(G^p_n - \{y^1_1, y^2_1, \ldots, y^p_1, x_1\})$. Therefore we get $\partial(G^p_n) = 2^p \partial(G^p_n - \{y^1_1, y^2_1, \ldots, y^p_1, x_1\}) + \partial(G^p_n - \{y^1_1, y^2_1, \ldots, y^p_1, x_1\}) = (2^p + 1)\partial(G^p_{n-1})$, because the graph $G^p_n - \{y^1_1, y^2_1, \ldots, y^p_1, x_1\}$ is isomorphic to G^p_{n-1}. □

Remark. By simple calculations we obtain the direct formula for $\partial(G^p_n)$ which is the following $\partial(G^n_p) = (2^p + 1)^n$.

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