FEKETE-SZEGÖ LIKE INEQUALITY FOR CERTAIN
SUBCLASSES OF ANALYTIC FUNCTIONS
RELATED TO COMPLEX ORDER

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Abstract: In the present investigation, the authors obtain Fekete-Szegö in-
equality for certain normalized analytic function \( f(z) \) defined on the open unit
disk for which

\[
1 + \frac{1}{b} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{\alpha} \left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} - 1 \right] < \phi(z)
\]

(\( 0 \leq \alpha \leq 1, b \neq 0 \), a complex number) lies in a region starlike with respect to
1 and symmetric with respect to the real axis. Also certain application of the
main result for a class of functions of complex order defined by convolution is
given. As a special case of this result, Fekete-Szegö inequality for a class of
functions defined through fractional derivatives is obtained.

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1. Introduction

Let $A$ denote the class of all analytic functions $f(z)$ of the form

$$ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \Delta := \{ z \in \mathbb{C} : |z| < 1 \}) $$

and $S$ be the subclass of $A$ consisting of univalent functions. Let $\phi(z)$ be an analytic function with positive real part on $\Delta$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\Delta$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Let $S^*(\phi)$ be the class of functions in $f \in S$ for which

$$ \frac{zf'(z)}{f(z)} \prec \phi(z) \quad (z \in \Delta) $$

and $C(\phi)$ be the class of functions in $f \in S$ for which

$$ 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z) \quad (z \in \Delta), $$

where $\prec$ denotes the subordination between analytic functions. These classes were introduced and studied by Ma and Minda [6]. They have obtained the Fekete-Szegö inequality for the functions in the class $C(\phi)$. Since $f(z) \in C(\phi)$ if and only if $zf'(z) \in S^*(\phi)$, we get the Fekete-Szegö problem for functions in the class $S^*(\phi)$. For a brief history of Fekete-Szegö problem for the class of starlike, convex and close-to-convex functions, see the recent paper by Srivastava et al [10].

Very recently Ravichandran et al [9] introduced the following classes of functions involving complex order.

**Definition 1.1.** Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on $\Delta$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\Delta$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in A$ satisfying

$$ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \phi(z). $$

The class $C_b(\phi)$ consists of functions $f \in A$ satisfying

$$ 1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \prec \phi(z). $$

They have obtained the Fekete-Szegö inequalities for functions in these classes.
Motivated by the aforementioned works, we obtain the Fekete-Szegö inequality for functions of complex order in a more general class $M_\alpha^b(\phi)$ of functions which we define below. Also we give applications of our results to certain functions defined through convolution (or the Hadamard product) and in particular we consider a class $M_\alpha^\lambda(\phi)$ of functions defined by fractional derivatives.

The motivation of this paper is to give a generalization of the Fekete-Szegö inequalities of subclasses of starlike and convex functions of complex order obtained by Ravichandran et al [9].

**Definition 1.2.** Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on $\Delta$ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk $\Delta$ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $M_\alpha^b(\phi)$ consists of all functions $f \in A$ satisfying

$$1 + \frac{1}{b} \left[ \left(1 + \frac{zf''(z)}{f'(z)}\right)^\alpha \left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} - 1 \right] \prec \phi(z) \quad (0 \leq \alpha \leq 1).$$

For fixed $g \in A$, we define the class $M_\alpha^b(\phi)$ to be the class of functions $f \in A$ for which $(f * g) \in M_\alpha^b(\phi)$.

Our definition of the function class $M_\alpha^b(\phi)$ is motivated essentially by the earlier investigation of Darus and Thomas [2], in which a closely related class can be found.

Also, we note that $M_0^1(\phi) = S_\alpha(\phi)$ and $M_1^1(\phi) = C_b(\phi)$.

To prove our main result, we need the following:

**Lemma 1.1.** (see [9]) If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part, then for any complex number $\mu$,

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\}$$

and the result is sharp for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2}, \quad p(z) = \frac{1 + z}{1 - z}.$$

**2. Fekete-Szegö Problem**

Our main result is the following:

**Theorem 2.1.** Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by
(1) belongs to $\mathcal{M}^\alpha_b(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2(1 + 2\alpha)} \max \left\{ 1, \frac{|B_2|}{B_1} + \left( \frac{2 + 7\alpha - \alpha^2}{2(1 + \alpha)^2} \right) b B_1 \right\}.$$  

The result is sharp.

**Proof.** If $f(z) \in \mathcal{M}^\alpha_b(\phi)$, then there is a Schwarz function $w(z)$, analytic in $\Delta$ with $w(0) = 0$ and $|w(z)| < 1$ in $\Delta$ such that

$$1 + \frac{1}{b} \left[ \left( 1 + \frac{zf''(z)}{f(z)} \right)^\alpha \left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} - 1 \right] = \phi(w(z)).$$  

(2)

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots.$$  

(3)

Since $w(z)$ is a Schwarz function, we see that $\Re(p_1(z)) > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by

$$p(z) := 1 + \frac{1}{b} \left[ \left( 1 + \frac{zf''(z)}{f(z)} \right)^\alpha \left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} - 1 \right] = 1 + b_1 z + b_2 z^2 + \cdots.$$  

(4)

In view of the equations (2), (3), (4), we have

$$p(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right)$$  

(5)

and from this equation (5), we obtain

$$b_1 = \frac{1}{2} B_1 c_1.$$  

(6)

and

$$b_2 = \frac{1}{2} B_1 (c_2 - \frac{1}{2} c_1^2) + \frac{1}{4} B_2 c_1^2.$$  

(7)

From the equation (4), we obtain

$$(1 + \alpha) a_2 = bb_1,$$

$$2 + 4\alpha) a_3 = bb_2 + \left[ \frac{2 + 7\alpha - \alpha^2}{2(1 + \alpha)^2} \right] a_2^2,$$

or equivalently we have

$$a_2 = \frac{bb_1}{1 + \alpha},$$  

(8)

$$a_3 = \frac{1}{2 + 4\alpha} \left[ bb_2 + \left( \frac{2 + 7\alpha - \alpha^2}{2(1 + \alpha)^2} \right) b^2 b_1^2 \right].$$  

(9)
Applying (6) in (8) and (6), (7) in (9), we have

\[ a_2 = \frac{bB_1c_1}{2(1 + \alpha)} , \]

\[ a_3 = \frac{bB_1c_2}{4(1 + 2\alpha)} + \frac{c_1^2}{8(1 + 2\alpha)} \left[ \frac{b^2B_1^2}{2} \left( \frac{2 + 7\alpha - \alpha^2}{(1 + \alpha)^2} \right) - b(B_1 - B_2) \right] . \]

Therefore we have

\[ a_3 - \mu a_2^2 = \frac{bB_1}{4(1 + 2\alpha)} \left\{ c_2 - vc_1^2 \right\} , \quad (10) \]

where

\[ v := \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \left( \frac{4\mu(1 + 2\alpha) - (2 + 7\alpha - \alpha^2)}{2(1 + \alpha)^2} \right) bB_1 \right] . \]

Our result now follows by an application of Lemma 1.1. The result is sharp for the function defined by

\[ 1 + \frac{1}{b} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\alpha \left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} - 1 \right] = \phi(z^2) \]

and

\[ 1 + \frac{1}{b} \left[ \left( 1 + \frac{zf''(z)}{f'(z)} \right)^\alpha \left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} - 1 \right] = \phi(z) . \] \]

For \( \alpha = 0 \), in Theorem 2.1 we get the result obtained by Ravichandran et al [9].

**Corollary 2.1.** Let \( \phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \). If \( f(z) \) given by (1) belongs to \( S^*_b(\phi) \), then

\[ |a_3 - \mu a_2^2| \leq \frac{B_1|b|}{2} \max \left\{ 1; \left| \frac{B_2}{B_1} + (1 - 2\mu)bB_1 \right| \right\} . \]

The result is sharp.

For a special case \( \alpha = 1 \), Theorem 2.1 reduces to another result obtained by Ravichandran et al [9].

**Corollary 2.2.** Let \( \phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots \). If \( f(z) \) given by (1) belongs to \( C_b(\phi) \), then

\[ |a_3 - \mu a_2^2| \leq \frac{B_1|b|}{6} \max \left\{ 1; \left| \frac{B_2}{B_1} + \left( 1 - \frac{3\mu}{2} \right) bB_1 \right| \right\} . \]

The result is sharp.

**Example 2.1.** By taking \( \alpha = 0 \), \( b = (1 - \beta)e^{-i\lambda} \cos \lambda \), \( \phi(z) = \frac{1 + \alpha}{1 - \alpha} \), we obtain the following sharp inequality for \( \lambda \)-spirallike function \( f(z) \) of order \( \beta \):
\[ |a_3 - \mu a_2^2| \leq \frac{(1 - \beta) \cos \lambda}{1 + \alpha} \times \max \left\{ 1, \left| e^{i\lambda} - \left[ \frac{4\mu(1 + 2\alpha) - (2 + 7\alpha - \alpha^2)}{(1 + \alpha)^2} \right] (1 - \beta) \cos \lambda \right| \right\}. \]

This result was obtained by Keogh and Merkes [5].

3. Application to Functions Defined by Fractional Derivatives

In order to introduce the class \( M_{b}^{\alpha,\lambda}(\phi) \), we need the following:

**Definition 3.1.** (see [8, 7]; see also [11, 12]) Let the function \( f(z) \) be analytic in a simply connected region of the \( z \)-plane containing the origin. The fractional derivative of \( f \) of order \( \lambda \) is defined by

\[
D_{\lambda}^zf(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_{0}^{z} f(\zeta) (z-\zeta)^{\lambda-1} d\zeta \quad (0 \leq \lambda < 1),
\]

where the multiplicity of \((z-\zeta)\) is removed by requiring \( \log(z-\zeta) \) is real for \((z-\zeta) > 0\).

Using the above Definition 3.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [8] introduced the operator \( \Omega_{\lambda} : A \rightarrow A \) defined by

\[
(\Omega_{\lambda} f)(z) = \Gamma(2-\lambda) z^{\lambda} D_{\lambda}^zf(z) \quad (\lambda \neq 2, 3, 4, \ldots).
\]

The class \( M_{b}^{\alpha,\lambda}(\phi) \) consists of functions \( f \in A \) for which \( \Omega_{\lambda} f \in M_{b}^{\alpha}(\phi) \). Note that \( M_{b}^{0,0}(\phi) = S_{b}^{*}(\phi) \) and \( M_{b}^{1,0}(\phi) = C_{b}(\phi) \). Also \( M_{b}^{\alpha,\lambda}(\phi) \) is the special case of the class \( M_{b}^{\alpha,\beta}(\phi) \) when

\[
g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^{n}. \quad (11)
\]

Let \( g(z) = z + \sum_{n=2}^{\infty} g_n z^n \) \((g_n > 0)\). Since \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in M_{b}^{\alpha,\beta}(\phi) \) if and only if \((f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in M_{b}^{\alpha}(\phi)\), we obtain the coefficient estimate for functions in the class \( M_{b}^{\alpha,\beta}(\phi) \), from the corresponding estimate for functions in the class \( M_{b}^{\alpha}(\phi) \).

Applying Theorem 2.1 for the function \((f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \ldots\), we get the following theorem after an obvious change of the parameter \( \mu \):

**Theorem 3.1.** Let the function \( \phi(z) \) be given by \( \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \). If \( f(z) \) given by (1) belongs to \( M_{b}^{\alpha,\beta}(\phi) \), then
The result is sharp. Since

\[ (\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n, \]

we have

\[ g_2 := \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda} \quad (12) \]

and

\[ g_3 := \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \quad (13) \]

For \( g_2 \) and \( g_3 \) given by (12) and (13), Theorem 3.1 reduces to the following:

**Theorem 3.2.** Let the function \( \phi(z) \) be given by \( \phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \). If \( f(z) \) given by (1) belongs to \( M_{b}^{\alpha,\lambda}(\phi) \), then

\[ |a_3 - \mu a_2^2| \leq \frac{(2-\lambda)(3-\lambda)B_1|b|}{12(1+2\alpha)} \times \max \left\{ 1, \left| \frac{B_2}{B_1} + \left[ \frac{(2+7\alpha-\alpha^2)(3-\lambda) - 6\mu(1+2\alpha)(2-\lambda)(3-\lambda)}{2(1+\alpha)^2(3-\lambda)} b B_1 \right] \right| \right\}. \]

The result is sharp.

References


