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SOME PROPERTIES OF SOLUTIONS OF NONLINEAR PARABOLIC PROBLEMS WITH NEUMANN CONDITIONS

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Abstract: We investigate the behavior of the solution of a nonlinear parabolic problem, when Neumann conditions are prescribed on the boundary $\partial\Omega \times [0, T]$, Ω a bounded R^N domain. We determine conditions on the geometry and data to insure a decay bound for the solution and its gradient.

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1. Introduction

Properties as blow up and decay bounds for solutions of linear and nonlinear parabolic problems under different boundary conditions have been well studied in the literature: we refer to the papers [2], [6], [7], [8], [9], with the references therein.

In this paper we consider a nonlinear parabolic problem which models a heat conduction process under Neumann boundary condition:

$$\begin{cases} \Delta u + f(|\nabla u|^2) = u_{,t}, & (\mathbf{x}, t) \in \Omega \times (0, T), \\ \dfrac{\partial u}{\partial n} = 0, & (\mathbf{x}, t) \in \partial\Omega \times [0, T], \\ u(\mathbf{x}, 0) = g(\mathbf{x}), & \mathbf{x} \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N with boundary $\partial\Omega \in C^{2+\epsilon}$, f is a non-

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negative and differentiable function with $f(0) = 0$. In (1.1) $\frac{\partial u}{\partial n}$ is the exterior normal derivative of u and the condition

$$\frac{\partial g}{\partial n} = 0 \tag{1.2}$$

is satisfied on $\partial\Omega$.

The particular case $f = 0$, when Ω is strictly convex and g has mean value zero on $\partial\Omega$, has been investigated in [6]; there the authors derived the following explicit decay bound (in time) for the solution u and its gradient, i.e.

$$|\nabla u|^2 + \alpha u^2 \leq \max_{\Omega}\{|\nabla g|^2 + \alpha g^2\}e^{-2\alpha t} \qquad \text{in } \Omega \times (t > 0), \tag{1.3}$$

with α a positive constant satisfying

$$0 \leq \alpha < \frac{\pi^2}{4D^2}, \tag{1.4}$$

D is the diameter of Ω.

In (1.1) the presence of the nonlinearity term f may determine the nonexistence of a solution for every $t > 0$, since the blow up phenomena may appear at a finite time T^*. If this is the case, we will consider (1.1) in the interval of existence $[0, T]$ with $T < T^*$ ([1], [4]). The purpose of the present paper is to investigate some proprieties of the solutions of problem (1.1): firstly we prove the boundedness of $|\nabla u(\mathbf{x}, t)|^2$ (Section 2) and we introduce conditions to preclude blow up (Section 3) and to obtain a decay bound of the solution. Then in Section 4, an exponential decay in time both for the solution and its gradient is obtained.

Throughout the paper we shall make use of the following notations: $u_{,i} := \frac{\partial u}{\partial x_i}$, $u_{,ik} := \frac{\partial^2 u}{\partial x_i \partial x_k}$, $k = 1, .., N$ and summation from 1 to N will be assumed on repeated indices as in $u_{,i}u_{,i} = \Sigma_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2 = |\nabla u|^2$ and $u_{,ii} = \Sigma_{i=1}^N \frac{\partial^2 u}{\partial x_i^2} = \Delta u$.

2. Boundedness of $|\nabla u|^2$

We firstly prove the following

Theorem 1. *Let $u(\mathbf{x}, t)$ be the classical solution of (1.1). Assume Ω strictly convex and*

$$\lim_{t\to 0} |\nabla u(\mathbf{x}, t)|^2 = |\nabla g(\mathbf{x})|^2. \tag{2.1}$$

Then

$$|\nabla u(\mathbf{x}, t)|^2 \leq \max_{\Omega} |\nabla g(\mathbf{x})|^2. \tag{2.2}$$

Proof. With standard computation we obtain

$$\Delta(|\nabla u|^2) - \frac{\partial |\nabla u|^2}{\partial t} + 2f'\nabla u\ \nabla(|\nabla u|^2) = 2u_{ik}u_{ik} \geq 0\,.$$

From (2.1) we deduce that $f'\nabla u$ is bounded, then we can apply Nirenberg maximum principle [5] and obtain that $\Phi(\mathbf{x}, t)$ takes its maximum value either:

i) at a boundary point $(\hat{\mathbf{x}}, \hat{t}) \in \partial\Omega \times [0, T]$;

ii) initially at a point $(\mathbf{x}, 0)$, $\mathbf{x} \in \Omega$.

We prove that the first possibility does not occur. Following Theorem 2 in [6], we can eliminate the first possibility: Let us assume that $\Phi := |\nabla u|^2$ takes its maximum value at a point $\hat{\mathbf{x}} \in \partial\Omega$ for some $\hat{t} \in [0, T]$. We assume that Φ is not constant. By the Friedman maximum principle (see [3]) we must have at $(\hat{\mathbf{x}}, \hat{t})$

$$\frac{\partial \Phi}{\partial n} > 0.$$

To exclude that the maximum is reached on $\partial\Omega$, we prove that, $\frac{\partial \Phi}{\partial n}(\hat{\mathbf{x}}, \hat{t}) \leq 0$. We have

$$\frac{\partial \Phi}{\partial n} = 2u_{,k}u_{,ki}n_i = 2u_{,k}(u_{,i}n_i)_{,k} - u_{,k}u_{,i}n_{i,k}, \quad x \in \partial\Omega, \quad 0 < t < T.$$

Since ∇u is tangent to $\partial\Omega$ and $\nabla(\frac{\partial u}{\partial n})$ is perpendicular to $\partial\Omega$, we have $u_{,k}(u_{,i}\ n_i)_{,k} = 0$ on $\partial\Omega$ and we obtain

$$\frac{\partial \Phi}{\partial n} = -2u_{,k}u_{,i}n_{i,k} = -2B_{ik}u_{,i}u_{,k} \leq 0. \tag{2.3}$$

In (2.3) B_{ik} is the curvature matrix associated with the surface $\partial\Omega$ (matrix positive semidefinite everywhere on the convex surface $\partial\Omega$). Following Theorem 2 (Corrigendum and Addendum) in [6], we conclude that Φ cannot attain its maximum value on $\partial\Omega$.

As a consequence Φ attains its maximum value initially at $t = 0$ and (2.2) is proved. □

3. A Decay Estimate for the Solution of (1.1)

Define

$$M := \frac{f(\Gamma^2)}{\Gamma^2}, \quad \Gamma^2 := \max_{\Omega} |\nabla g(\mathbf{x})|^2. \tag{3.1}$$

Firstly we prove the following:

Lemma 1. *Let u be the solution of (1.1). Assume*

$$\frac{d}{ds}\Big(\frac{f(s)}{s}\Big) \geq 0, \tag{3.2}$$

then

$$\Delta u + M|\nabla u|^2 - u_{,t} \geq 0 \text{ in } \Omega \times [0,T], \tag{3.3}$$

with M in (3.1).

Proof. Since from (3.2) $\frac{f(s)}{s}$ is non decreasing, from (2.2) we have $\frac{f(|\nabla u|^2)}{|\nabla u|^2} \leq \frac{f(\Gamma^2)}{\Gamma^2}$. By (3.1) we have

$$0 = \Delta u + \frac{f(|\nabla u|^2)}{|\nabla u|^2}|\nabla u|^2 - u_{,t} \leq \Delta u + M|\nabla u|^2 - u_{,t}. \qquad \square \tag{3.4}$$

To obtain a decay estimate for u, we introduce the auxiliary function

$$v(\mathbf{x},t) = e^{Mu(\mathbf{x},t)} - 1, \quad (\mathbf{x},t) \in \Omega \times [0,T]. \tag{3.5}$$

Denote for brevity by h($\mathbf{x}$): $= v(\mathbf{x},0) = (e^{Mg(\mathbf{x})} - 1)$. We prove the following:

Theorem 2. *Assume Lemma 1 holds and g satisfies*

$$\int_\Omega h(\mathbf{x})dx = 0. \tag{3.6}$$

Then

$$u(\mathbf{x},t) \leq \frac{2\pi}{MD}\tilde{H}e^{-\frac{\pi^2}{2D^2}t}, \tag{3.7}$$

with D diameter of Ω and $\tilde{H}^2 = \max_\Omega\{|\nabla h|^2 + \frac{\pi^2}{4D^2}h^2\}$.

Proof. As a consequence of (3.3) the function v satisfies the parabolic problem

$$\begin{cases} \Delta v - v_{,t} \geq 0, & (\mathbf{x},t) \in \Omega \times [0,T], \\ \frac{\partial v}{\partial n} = 0, & (\mathbf{x},t) \in \partial\Omega \times [0,T], \\ v(\mathbf{x},0) = h(\mathbf{x}), & \mathbf{x} \in \Omega. \end{cases} \tag{3.8}$$

If $\tilde{v}(\mathbf{x},t)$ is the solution of the problem (3.8) with equality sign in the first equation, by a Classical Comparison Theorem (see [10]), we have $v \leq \tilde{v}$. The hypothesis (3.6) for h allows us to use for $\tilde{v}$ Theorem 4 in [6] with $\alpha \to \frac{\pi^2}{4D^2}$ and to obtain

$$|\tilde{v}| \leq \frac{2D}{\pi}\tilde{H}e^{-\frac{\pi^2}{2D^2}t}.$$

By (3.5) we conclude that

$$|u(\mathbf{x},t)| \leq \frac{1}{M}\ln\Big[\frac{2\pi}{D}\tilde{H}e^{-\frac{\pi^2}{2D^2}t} + 1\Big] \leq \frac{1}{M}\frac{2\pi}{D}\ \tilde{H}\ e^{-\frac{\pi^2}{2D^2}t}. \qquad \square \tag{3.9}$$

4. Decay Bounds in Time for u and $|\nabla u|$

Let us introduce the auxiliary function

$$\Psi(\mathbf{x},t) = \left\{|\nabla u|^2 + \beta u^2\right\} e^{2\beta t}, \tag{4.1}$$

where β is a positive constant. By using a maximum principle (see [10]) we derive in this section an exponential decay bound for u and $|\nabla u|$ in $\Omega \times [0,T]$.

Theorem 3. *Let $u(\mathbf{x},t)$ be the solution of (1.1) with $g(\mathbf{x}) \geq 0$ and f be assumed to satisfy*

$$\frac{d}{ds}\Big(\frac{f(s^2)}{s}\Big) \geq 0. \tag{4.2}$$

Then the function $\Psi(\mathbf{x},t)$ defined in (4.1) satisfies the parabolic inequality

$$\Delta\Psi + |\nabla u|^{-2} W_k \Psi_{,k} - \Psi_{,t} \geq 0, \tag{4.3}$$

with $W_k = 2\beta u u_{,k}$.

Furthermore if Ω is a bounded strictly convex domain, the following decay estimate holds

$$\left\{|\nabla u|^2 + \beta u^2\right\} \leq \mathcal{H} e^{-2\beta t}, \quad (\mathbf{x},t) \in \Omega \times [0,T], \tag{4.4}$$

where $\mathcal{H} = \max\left\{\mathcal{H}_0, \mathcal{H}_1\right\}$, with

$$\mathcal{H}_0 = \max_{\Omega}\{|\nabla g|^2 + \beta g^2\}, \quad \mathcal{H}_1 = \Psi(\bar{\mathbf{x}},\bar{t}), \tag{4.5}$$

with $(\bar{\mathbf{x}},\bar{t})$ a critical point of u.

Proof. We firstly observe that if $g \geq 0$, as consequence of the maximum principle, we have $u \geq 0$. By differentiation of (4.1) we obtain

$$\Psi_{,k} = 2e^{2\beta t}\left\{u_{,i}u_{,ik} + \beta u u_{,k}\right\}, \tag{4.6}$$

$$\Delta\Psi = 2\ e^{2\beta t}\left\{u_{,ik}u_{,ik} + u_{,i}\Delta u_{,i} + \beta|\nabla u|^2 + \beta u \Delta u\right\}.$$

By using the equation in (1.1) we have

$$2\ e^{2\beta t}\left\{u_{,ik}u_{,ik} + u_{,i}u_{,it} - 2f'(|\nabla u|^2)u_{,i}u_{,k}u_{,ik} + \beta|\nabla u|^2 \right.$$
$$\left. + \beta u u_{,t} - \beta u f(|\nabla u|^2)\right\},$$

$$\Psi_{,t} = 2\ e^{2\beta t}\left\{u_{,i}u_{,it} + \beta u u_{,t} + \beta|\nabla u|^2 + \beta^2 u^2\right\},$$

$$\Delta\Psi - \Psi_{,t} \tag{4.7}$$
$$= 2e^{2\beta t}\left\{u_{,ik}u_{,ik} - 2f'(|\nabla u|^2)u_{,i}u_{,k}u_{,ik} - \beta u f(u) - \beta^2 u^2\right\}.$$

To estimate the term $u_{,ik}u_{,ik}$ we firstly observe that from (4.6) we have

$$u_{,i}u_{,ik}u_{,j}u_{,jk} \geq \beta^2 u^2 |\nabla u|^2 - \beta u u_{,k}\Psi_{,k}e^{-2\beta t}.$$

Then, by using the Schwarz inequality, we obtain

$$u_{,ik}u_{,ik} \geq |\nabla u|^{-2}u_{,i}u_{,ik}u_{,j}u_{,jk} \geq \beta^2 u^2 - \frac{1}{2}|\nabla u|^{-2}W_k\Psi_{,k}e^{-2\beta t}, \tag{4.8}$$

with $W_k = 2\beta u u_{,k}$. By replacing (4.8) in (4.7), we obtain

$$\Delta\Psi + |\nabla u|^{-2}W_k\Psi_{,k} - \Psi_{,t} \geq 2e^{2\beta t}\beta u\Big\{2f'(|\nabla u|^2)|\nabla u|^2 - f(|\nabla u|^2)\Big\}. \tag{4.9}$$

Now by inserting (4.2) in (4.9), we obtain (4.3). To prove (4.4)-(4.5) we observe that by the Nirenberg maximum principle [5], $\Psi(\mathbf{x},t)$ takes its maximum value either

i) at a boundary point $(\hat{\mathbf{x}},\hat{t}) \in \partial\Omega \times [0,T]$;

ii) at an interior critical point of $u(\mathbf{x},t)$, $(\bar{\mathbf{x}},\bar{t}) \in \Omega\times[0,T]$ with $\nabla u(\bar{\mathbf{x}},\bar{t}) = 0$;

iii) initially at a point $(\mathbf{x},0)$, $\mathbf{x} \in \Omega$, i.e.

$$\Psi(\mathbf{x},t) \leq \max\begin{cases} \Psi(\hat{\mathbf{x}},\hat{t}), \quad (\hat{\mathbf{x}},\hat{t}) \in \partial\Omega \times [0,T], \\ \Psi(\bar{\mathbf{x}},\bar{t}) \text{ with } \nabla u(\bar{\mathbf{x}},\bar{t}) = 0 \quad \bar{\mathbf{x}} \in \Omega, \\ \max_\Omega[|\nabla g|^2 + \beta g^2], \end{cases}$$

with $g(\mathbf{x}) = u(\mathbf{x},0)$.

Now by using the strictly convexity of Ω we show that (i) cannot hold. Following Theorem 2 (Corrigendum and Addendum) in [6] we obtain that the auxiliary function $\Psi(\mathbf{x},t)$ cannot attain its maximum value at a point $\hat{\mathbf{x}} \in \partial\Omega$, for some $\hat{t} \in [0,T]$, unless $g = 0$. In fact if Ψ attains its maximum value in $\hat{\mathbf{x}} \in \partial\Omega$ for some $\hat{t} \in [0,T]$, by the boundary maximum principle [3] we must have

$$\frac{\partial\Psi}{\partial n}(\hat{\mathbf{x}},\hat{t}) > 0.$$

On the contrary we have

$$\frac{\partial\Psi}{\partial n}(\hat{\mathbf{x}},\hat{t}) = -2B_{ik}u_{,i}u_{,ik} \leq 0,$$

where B_{ik} is the the curvature matrix associated with the surface $\partial\Omega$, (matrix positive semidefinite everywhere on the strictly convex surface $\partial\Omega$).

Then

$$\Psi(\mathbf{x},t) \leq \max\begin{cases} \Psi(\bar{\mathbf{x}},\bar{t}) \text{ with } \nabla u(\bar{\mathbf{x}},\bar{t}) = 0 \quad \bar{\mathbf{x}} \in \Omega, \\ \max_\Omega[|\nabla g|^2 + \beta g^2]. \end{cases}$$

Then (4.4)-(4.5) hold and Theorem 3 is proved. □

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