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ON VECTORIAL INNER PRODUCT SPACES

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1. Preliminaries

Definition 1.1. An ordered vector space X over $I\!R$ is called a *Riesz space* (or a vector lattice) if any doubleton $\{x, y\} \subset X$ has a least upper bound $\sup(x, y)$ and a greatest lower bound $\inf(x, y)$.

Definition 1.2. The ordering of X is called *Archimedean* if $\boldsymbol{nx} \preceq \boldsymbol{y}$ for some pair $\boldsymbol{(x, y)}$ and all $\boldsymbol{n} \in I\!N$ implies $\boldsymbol{x} \preceq \boldsymbol{0}$.

Definition 1.3. A Riesz space X is called *Dedekind complete* (or order complete) if each non-empty subset bounded from above possesses a least upper bound.

Definition 1.4. In a Riesz space X if there exists a positive element $\boldsymbol{e}$ such that for every $x \in X$ there exists a positive real number $\boldsymbol{\alpha_x}$ satisfying $|\boldsymbol{x}| \preceq \boldsymbol{\alpha_x e}$, the Riesz space is called *unitary*. The element $\boldsymbol{e}$ is called a *unit* (or a strong unit).

Definition 1.5. A *unitary Archimedean Riesz space* is called a *Yosida space*.

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Definition 1.6. *Ideals* J of a Riesz space are linear subspaces characterized by the property: $\boldsymbol{((x \in J\ ,\ y \in X\ \wedge\ |y| \preceq |x|) \Rightarrow y \in J)}$.

Let X be a unitary Riesz space ($\boldsymbol{e}$ a fixed unit), J an arbitrary fixed maximal ideal and X/J the quotient space of all equivalence classes modulo J.

The quotient space X/J consists of all real multiples of $\boldsymbol{e}$, that is, given $x \in X$, there exists a real number λ such that $[x] = \lambda[\boldsymbol{e}]$. We will denote this number by $x(J)$. For x fixed and J running through the set $\mathcal{M}$ of all maximal ideals in X, we thus obtain a bounded real-valued mapping defined on $\mathcal{M}$.

Definition 1.7. *Bands* are ideals with the additional property that they contain the suprema of arbitrary subsets, whenever these suprema exist in X.

Definition 1.8. In a Riesz space a maximal band is called a *hypermaximal band* if it is also a maximal ideal.

Definition 1.9. A Yosida space, Dedekind complete and such that the intersection of all its hypermaximal bands is the zeroelement of the space is called a *$\mathcal{B}$-regular Yosida space* ($\mathcal{B}$ denotes the set of all hypermaximal bands).

A linear mapping h from the Riesz space X_1 into the Riesz space X_2 is called a *Riesz homomorphism* provided $\inf(h(x), h(y)) = 0$ holds for every pair of elements $x, y \in X_1$ satisfying $\inf(x, y) = 0$. If, in addition, h is bijective, then it is called a *Riesz isomorphism* and the spaces X_1 and X_2 are said to be *Riesz isomorphic.*

Theorem 1.1. (Yosida Theorem) *Let Y be a $\mathcal{B}$-regular Yosida space. The mapping:* $\boldsymbol{(x \in Y) \longrightarrow (J \in \mathcal{B} \longrightarrow x(J) \in \mathbb{R})}$, *defines a Riesz isomorphism from Y onto the Riesz space $B(\mathcal{B})$ of all bounded real-valued mappings defined on $\mathcal{B}$. Moreover, we have*

1. $x(J) = 0$ *iff* $x \in J$, $x = 0$ *iff* ($x(J) = 0$, *for all* $J \in \mathcal{B}$) *2.* $(x + y)(J) = x(J) + y(J), (\alpha x)(J) = \alpha x(J)$ *3.* $x \preceq y$ *iff* ($x(J) \leq y(J)$, *for all* $J \in \mathcal{B}$) *4.* $(|x|)(J) = |x(J)|$ *5.* $\boldsymbol{e}(J) = 1$, *for all* $J \in \mathcal{B}$ *6.* $((\sup(x, y))(J) = \max(x(J), y(J)), (\inf(x, y))(J) = \min(x(J), y(J)))$ *7. If* $J_1 \neq J_2$ *there exists* $x \in Y$ *such that* $x(J_1) \neq x(J_2)$.

In view of this theorem, given any $\mathcal{B}$-regular Yosida space Y it seems desirable to assume, without loss of generality, that Y is the space of all bounded real-valued mappings on a certain set A and we will denote it by $B(A)$.

In the $\mathcal{B}$-regular Yosida space $B(A)$ of all bounded real-valued mappings on a certain set A we have

$$\boldsymbol{\mathcal{B} = \{J_\alpha : \alpha \in A\}}, \quad \text{where} \quad \boldsymbol{J_\alpha = \{x \in B(A) : x(\alpha) = 0\}}.$$

2. Vectorially Normed Spaces

A *vectorial norm* p is a mapping from E into $B(A)$ with the properties of a usual norm, i.e.

$$\boldsymbol{p(\lambda u) = |\lambda| p(u), \quad p(u+v) \preceq p(u) + p(v), \quad (p(u) = 0 \Rightarrow u = 0)}.$$

The space E with a vectorial norm is named a *vectorially normed space.*

Let p be a vectorial norm from the linear space E into $B(A)$, and let us consider the family $(\theta_\alpha)_{\alpha\in A}$ of usual seminorms defined in the following way:

For each $\alpha \in A$, we set

$$\boldsymbol{\theta_\alpha(u) := (p(u))(\alpha) \quad \forall u \in E}.$$

The kernel of θ_α will be denoted by V_α, i.e.

$$\boldsymbol{V_\alpha = \{u \in E : (p(u))(\alpha) = 0\}}.$$

For each $\alpha \in A$, we shall also consider the subspace

$$\boldsymbol{W_\alpha := \{u \in E : (p(u))(\beta) = 0, \beta \neq \alpha\}}.$$

For each finite subset S of A, we define $W(S)$ as the direct sum

$$\boldsymbol{W(S) := \bigoplus\nolimits_{\alpha\in S} W_\alpha},$$

we also define

$$\boldsymbol{W := \bigcup\nolimits_{S\in\mathcal{PF}(A)} W(S)}$$

($\mathcal{PF}(A)$ denotes the set of all finite subsets of A).

The $\mathcal{B}$-regular Yosida space $B(A)$ is a Banach lattice with the usual norm defined by

$$\boldsymbol{||x|| := \sup\nolimits_{\alpha\in A} |\,x\,(\alpha)|}.$$

Let us also consider the usual norm g defined on E by

$$\boldsymbol{g(u) := ||p(u)|| = \sup\nolimits_{\alpha\in A}(p(u)(\alpha))}.$$

Definition 2.1. Let p be a vectorial norm defined on a linear space E and with range in $(B(A), ||.||)$. The vectorial norm p is said to be *regular* if $\overline{W} = E$ (considering in E the topology induced by the norm $g(.) = ||p(.)||$).

The equivalence of norms in Banach lattices allows us to set forth the last definition for a vectorial norm with range in any Banach lattice.

In what follows we denote by $\mathcal{PN}(A)$ the set of all finite or countably infinite subsets of A.

Now we state the following theorem for purposes of later reference.

Theorem 2.1. (Representation Theorem) *Let E be a real or a complex linear space, p a regular vectorial norm defined on E with its range in $(B(A), ||.||)$. Let us also suppose that $(E, g(.))$ is a Banach space. Then:*

1. $E = V_\alpha \oplus W_\alpha \qquad \forall \alpha \in A$.

2. Let P_α be the projection of $E = V_\alpha \oplus W_\alpha$ onto W_α. For each $u \neq 0$ in E there exists a set $S_u \in \mathcal{PN}(A)$ such that $P_\alpha u = u_\alpha \neq 0$ if and only if $\alpha \in S_u$. The family $\{u_\alpha\}_{\alpha \in S_u}$ is summable (with respect to the topology induced by the norm $g(.)$), with the sum u, i.e.

$$\boldsymbol{u = \sum_{\alpha \in S_u} u_\alpha, \quad u_\alpha \in W_\alpha, \quad u_\alpha \neq 0, \quad \forall \alpha \in S_u.}$$

Moreover, the representation of the element u in the above form is unique.

3. Vectorial Inner Product Spaces

We will say that a real linear space E is a vectorial inner product space associated to $B(A)$ if there is a mapping F from $E \times E$ with range in $B(A)$, subject to the following axioms:

A1. $\boldsymbol{F(u,v) = F(v,u)}$, A2. $\boldsymbol{F(u+v,w) = F(u,w) + F(v,w)}$,

A3. $\boldsymbol{F(ku,v) = kF(u,v)}$, A4. $\boldsymbol{F(u,u) \succeq 0}$ and $\boldsymbol{F(u,u) = 0}$ if and only if $\boldsymbol{u = 0}$,

where u, v, w are arbitrary elements of E and k is an arbitrary real scalar. The mapping F will be called a vectorial inner product.

Let us define, for each $\alpha \in A$, the mapping

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{R}_0^+ \,, \\ u & \longrightarrow & (F(u,u)(\alpha))^{1/2} \,. \end{array}$$

The mappings just defined are usual seminorms, moreover

$$(F(u,u)(\alpha))^{1/2} = 0 \ \text{ if and only if } \ F(u,u) \in J_\alpha$$

and, for all $\alpha \in A$,

$$|\boldsymbol{F(u,v)(\alpha)}| \leq \boldsymbol{(F(u,u)(\alpha))^{1/2}.(F(v,v)(\alpha))^{1/2}} \quad \forall \boldsymbol{u, v \in E}.$$

Let us now consider the usual product of bounded real-valued mappings defined on $B(A)$. The mapping

$$\begin{array}{lccc} p: & E & \longrightarrow & B(A), \\ & u & \longrightarrow & p(u) = F(u,u)^{1/2} \end{array}$$

is a vectorial norm. The last inequality can be written in the form

$$|\boldsymbol{F(u,v)}| \preceq \boldsymbol{p(u)p(v)}.$$

We assume that in E the topology of a normed linear space is given, using the norm, $g(u) = ||p(u)|| = ||(F(u,u))^{1/2}||$.

Definition 3.1. Let E be a linear space, F a vectorial inner product from $E \times E$ into the space $(B(A), ||.||)$ and suppose that $p(.) := (F(.,.)^{1/2}$ is a regular vectorial norm. The space E is said to be a vectorial Hilbert space if it is a complete space (with respect to the topology induced by the norm $g(.) = ||F(.,.)^{1/2}||$).

3.1. Orthogonality and Orthonormal Sets

Let E be a vectorial inner product space associated to $B(A)$.

We say that $u, v \in E$ are orthogonal, if $F(u,v) = 0$ (symbolically $u \perp v$).

Let us define the following elements of $B(A)$. For each $\alpha \in A$ the element $e_\alpha \in B(A)$ is such that $e_\alpha(\alpha) = 1$ and $e_\alpha(\beta) = 0$ if $\beta \neq \alpha$.

Definition 3.1.1. A set S of elements of E is called an orthogonal set if $u \perp v$ for every pair u, v for which $u \in S$, $v \in S$, and $u \neq v$. If in addition for every $u \in S$ there exists $\alpha_u \in A$ such that $F(u,u) = e_{\alpha_u}$, the set S is called an orthonormal set.

Definition 3.1.2. Let S be an orthonormal set in the space E. The set S is called maximal if there exists no orthonormal set of which S is a proper subset. The set S is called complete if $F(v,u) = 0$ for all $u \in S$ implies $v = 0$.

Theorem 3.1.1. *Let E be a vectorial Hilbert space associated to $B(A)$ and S an orthonormal set in E. Then S is maximal if and only if S is complete.*

4. The Space $\overline{W_{B(A)}}$

As can be easily observed the norm complete and $\mathcal{B}$-regular Yosida space $(B(A), ||.||)$, with the usual product of bounded real-valued mappings, is a commutative normed Yosida algebra (i.e., it is a Yosida space with a product satisfying $xy = yx$, $(xy)z = x(yz)$, $x(y+z) = xy + xz$, $(kx)(y) = k(xy)$, if $0 \preceq w$ and $x \preceq y$ then $wx \preceq wy$, and $||xy|| \leq ||x||\,||y||$). Ideals J of the Yosida algebra are ideals of the Riesz space $B(A)$ with the additional property: $y \in J \Rightarrow (\forall x \in B(A), xy \in J)$. The Yosida algebra is still $\mathcal{B}$-regular since $\bigcap_{J \in \mathcal{B}} J = \{0\}$, where $\mathcal{B}$ is the set of all hypermaximal bands of the algebra.

The algebra $B(A)$ can also be regarded as a vectorially normed space with the vectorial norm $m(x) = \sup(x, -x) = |x|$ with its range in $B(A)$.

We must observe that $B(A)$ as a vectorially normed space is also a Banach space with the norm

$$\boldsymbol{g(x) = \sup_{\alpha \in A}((m(x))(\alpha) = ||x||}.$$

In this particular case we have that, given $\alpha \in A$ and $x \in B(A)$, we have

$$\boldsymbol{\theta_\alpha(x) = (m(x))(\alpha) = |x(\alpha)| \quad , \quad V_\alpha = J_\alpha \quad , \quad W_\alpha = J_\alpha^\perp}.$$

Let $W_{B(A)}$ denote the linear subspace of $B(A)$ of all bounded real-valued mappings with finite support. We define the space $\overline{W_{B(A)}}$ as the closure of $W_{B(A)}$ with respect to the topology induced by the norm $||.||$.

Theorem 4.1. *Let $(B(A), ||.||)$ be the $\mathcal{B}$-regular and norm complete Yosida algebra of all bounded real-valued mappings defined on A. Consider also the vectorially normed space $(B(A), m = |.|)$. Then:*

1. For each $x \neq 0$ in $\overline{W_{B(A)}}$ there exists a set $S_x \in \mathcal{PN}(A)$ such that $x(\alpha) = 0$ if and only if $\alpha \notin S_x$. The family $\{x(\alpha)e_\alpha\}_{\alpha \in S_x}$ is summable (with respect to the topology induced by the norm $||.||$) with the sum x, i.e. $x = \sum_{\alpha \in S_x} x(\alpha)e_\alpha$. Conversely, if an element $x \in B(A)$ is of the form $x = \sum_{\alpha \in S} k_\alpha e_\alpha$, $S \in \mathcal{PN}(A)$, then $x \in \overline{W_{B(A)}}$ and $k_\alpha = x(\alpha)$.

2. The linear subspace $\overline{W_{B(A)}}$ (as a subset of the norm complete Yosida algebra $B(A)$) is a norm complete Archimedean and Dedekind complete Riesz algebra.

5. Vectorial Hilbert Spaces

In what follows $(E, g = ||p(.)||)$ is a vectorial Hilbert space with $p(.) = (F(.,.))^{1/2}$, where F is a vectorial inner product defined on $E \times E$ and with its range in the $\mathcal{B}$-regular norm complete Yosida algebra $(B(A), ||.||)$.

5.1. Bessel Inequality and Parseval Identity in Vectorial Hilbert Spaces

Theorem 5.1.1. (Bessel Inequality – Countably Infinite Case) *Let* $(w_n)_{n \in I\!N}$ *be an orthonormal sequence of elements of* E. *Given* $u, v \in E$ *the series* $\sum_{i=1}^{\infty} |F(u, w_i)F(v, w_i)|$ *is convergent in* $\overline{W_{B(A)}}$ *and we have*

$$\sum_{i=1}^{\infty} |F(u, w_i)F(v, w_i)| \preceq p(u)p(v).$$

Theorem 5.1.2. (Bessel Inequality – General Case) *Let* S *be an arbitrary orthonormal set in* E. *The set of* $w \in S$ *such that* $F(u, w) \neq 0$ *(*u *any fixed element of* E*) is either finite or countably infinite. Given* $u, v \in E$ *we have that* $\sum_{w \in S} |F(u, w)F(v, w)|$ *defines an element in* $\overline{W_{B(A)}}$, *and*

$$\sum_{w \in S} |F(u, w)F(v, w)| \preceq p(u)p(v)$$

it being understood that the sum on the left includes all $w \in S$ *for which* $F(u, w)F(v, w) \neq 0$, *and is, therefore, either a finite series or a convergent series with a countable infinity of terms.*

Theorem 5.1.3. (Parseval Identity) *An orthonormal set* S *is maximal if and only if*

$$p^2(u) = \sum_{w \in S} (F(u, w)(\alpha_w))^2 e_{\alpha_w}, \qquad \forall u \in E.$$

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