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COMPETITIVE LOGISTIC NETWORKS WITH ADAPTATION

A. Raimondi[1], C. Tebaldi[2 §]

[1,2]DIMAT - Dipartimento di Matematica
Politecnico di Torino
24, Corso Duca degli Abruzzi, Torino, 10129, ITALY
[1]e-mail: alessia.raimondi@polito.it
[2]e-mail: claudio.tebaldi@polito.it

Abstract: We consider a network where the local activity level of each node is described by the logistic equation and the interaction among nodes take the form of competition, also including a mechanism of adaptation. We study here analytically the equilibria of such system, depending on the interaction strength and the size of the network, and we prove existence of classes of invariant subspaces which allow the introduction of a reduced model, where n appears as parameter. One of such models, of four equations, gives complete account of all equilibria in the complete system.

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1. Introduction

A network is a collection of parts with connecting links for which interactions among the parts give rise to collective behavior of the system. The concept has found applications in broader and broader areas of complex systems, having the Wold Wide Web as a kind of paradigm, see for example Barabasi [1].

We consider a network as a set of n nodes whose local activity level is governed by the logistic equation. Interactions among nodes take the form of competition which, through past history dependence, also includes adaptation, a characteristic feature of all complex systems. As a consequence the dynamics

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§Correspondence author

of the network is governed by a system of n^2 ordinary differential equations.

We show the existence of invariant subspaces which allow the introduction of reduced models, where n appears as a parameter, and we make use of those to study analytically the equilibria of the system in dependence of the relevant parameters, namely the strength of competition, the adaptation rate and the network size.

2. The Model

We consider a general network of n interconnected nodes, in which the dynamics of a node is described by the logistic equation

$$\frac{dM_i(t)}{dt} = r_i\big[1 - \frac{M_i(t)}{k_i}\big]M_i(t)\,, \tag{1}$$

where the positive variable $M_i(t)$ denotes the activity level of node i at time t and the positive parameters r_i and k_i are, respectively, the intrinsic growth rate and the time asymptotic value of the activity level of node i. Without loss of generality, by a suitable change of variables, in (1) we can suppose $k_i = 1$. Having as main aim the discussion of the role of interactions among nodes, we suppose for all them the same intrinsic growth rate, $r_i = r$, and we can assume $r = 1$ rescaling time.

The interactions among nodes take place in the form of competition: for each node i we have, in (1), the additional term

$$-\sum_{\substack{j=1\\ j\neq i}}^{n} \beta_{ij} M_i M_j\,, \qquad 1 \le i \le n\,,$$

where β_{ij} is a positive function that represents the strength of interaction of node j to node i.

In this work we want to consider also a mechanism of adaptation in competition, therefore we consider β_{ij} proportional to the level of activity of nodes j and i over the past

$$\beta_{ij}(t) = C\int_{-\infty}^{t} M_i(u)M_j(u)K_{T_j}(t-u)du\,, \quad 1 \le i,j \le n, j \neq i\,, \tag{2}$$

where $K_T(t) = e^{(-t/T)}/T$ is a delay kernel representing a short term memory effect, see Noonburg [5]. T and C define the adaptation rate of the interaction and its strength which, together with the dimension of the network n, are the relevant parameters.

Differentiating β_{ij} with respect to t, (2) can be replaced by

$$\frac{d\beta_{ij}}{dt} = \frac{CM_iM_j - \beta_{ij}}{T_j},$$

see Lacitignola and Tebaldi [4] for details. As a first step, here we consider also the same adaptation rate for all nodes, $T_j = T$.

The dynamics of the network is therefore described by the following set of n^2 ordinary differential equation

$$\left\{ \begin{array}{lcl} \frac{dM_i}{dt} & = & [1 - M_i]M_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} \beta_{ij} M_i M_j \,, \\ \frac{d\beta_{ij}}{dt} & = & \frac{CM_iM_j - \beta_{ij}}{T}\,, \quad 1 \leq i,j \leq n, i \neq j\,, \end{array} \right. \tag{3}$$

where the first n equations provide the activity level for each node, while the following $(n^2 - n)$ define adaptive competition.

3. Reduced Models

Since we want to study analytically system (3) in full generality, we have found useful to start considering the invariant subspaces related to symmetries.

System (3) is found to admit the invariant subspace

$$\beta_{ji} = \beta_{ij}\,, \quad 1 \leq i,j \leq n, j \neq i, \tag{4}$$

for which it is easy to see that it is also attractive. In subspace (4) we obtain a system of $(n^2 + n)/2$ differential equations for the M_i's and for the β_{ij} with $j > i$. Because of the attractivity of (4), such system gives complete account of all the attractors in system (3).

Furthemore, for (3) we have the following invariant subspace

$$M_i = M, \quad \beta_{ij} = \beta\,, \quad 1 \leq i,j \leq n, j \neq i. \tag{5}$$

In this subspace system (3) becomes

$$\left\{ \begin{array}{lcl} \frac{dM}{dt} & = & [1 - M]M - (n-1)\beta M\,, \\ \frac{d\beta}{dt} & = & \frac{CM^2 - \beta}{T}\,, \end{array} \right. \tag{6}$$

where n appears as a parameter.

We can also choose arbitrarily $1 \leq h,k,l \leq n$ with $k,l \neq h$ and $k \neq l$, and we have the following invariant subspace

$$M_i = M_k\,, \quad i \neq h;\ \beta_{ij} = \beta_{hk}\,, \quad i = h \vee j = h;\ \beta_{ij} = \beta_{kl}\,, \quad i,j \neq h\,, \tag{7}$$

of which (5) is a subspace. In (7), system (3) becomes

$$\begin{cases} \dfrac{dM_h}{dt} &= M_h[1 - M_h - (n-1)\beta_{hk}M_k]\,, \\ \dfrac{dM_k}{dt} &= M_k[1 - M_k - \beta_{kh}M_h - (n-2)\beta_{kl}M_l]\,, \\ \dfrac{d\beta_{hk}}{dt} &= \dfrac{CM_hM_k - \beta_{hk}}{T}\,, \\ \dfrac{d\beta_{kl}}{dt} &= \dfrac{CM_k^2 - \beta_{kl}}{T}\,, \end{cases} \tag{8}$$

where n appears as a parameter.

In the same line, if we choose $1 \leq h, k, l, t \leq n$ with $k, l, t \neq h$, $l, t \neq k$ and $t \neq l$ and n_h, n_k, n_l such that $n_h + n_k + n_l = n$, we have the following invariant subspace:

$$\begin{aligned} M_i &= M_h, \quad i = 1, \ldots, n_h\,, \\ M_i &= M_k, \quad i = n_h + 1, \ldots, n_h + n_k\,, \\ M_i &= M_l, \quad i = n_h + n_k + 1, \ldots, n\,, \\ \beta_{ij} &= \beta_{hk}, \quad i = h \vee j = h\,, \\ \beta_{ij} &= \beta_{hl}, \quad i = l \vee j = l, i, j \neq k\,, \\ \beta_{ij} &= \beta_{hl}, \quad i = l \vee j = l, i, j \neq k\,, \\ \beta_{ij} &= \beta_{lt}, \quad i \neq h, k\,, \end{aligned} \tag{9}$$

of which (5) and (7) are subspaces.

Existence of the invariant subspaces, described above, turns out to be essential for the analytic discussion of the equilibria in the model and their stability properties.

4. Equilibria

The equilibria for system (3) have $\beta_{ij} = CM_iM_j$ and the M_i's are solutions of $M_i - M_i^2 - \sum_{\substack{j=1 \\ j\neq i}}^{n} CM_i^2M_j^2 = 0$ $i = 1, \ldots, n$. Therefore, in the following, the equilibria will be characterized by the variables M_i's only, for the sake of conciseness. We notice that the adaptation parameter T does not play any role for the existence of the equilibria.

We call interior equilibria the ones with positive M_i's.

In the logistic case, $C = 0$, we obtain: the interior equilibrium R_0 such that $M_i = 1,$ for $i = 1, \ldots, n$ and the class of equilibria S_0 which have one or more $M_i = 1$ replaced by $M_i = 0$. It is easy to show that R_0 is stable and the S_0's are unstable for any T and n.

For $C \neq 0$, equilibria in subspace (5) have to satisfy $(n-1)CM^3+M-1=0$, since $\beta = CM^2$. The cubic equation has one and only one positive root r for any $C > 0$ and for any $n \geq 3$. One equilibrium $R = (r)$ exists. Consequently, for the complete model (3) an equilibrium $R = (r, r, \ldots, r)$ exists for any n.

In subspace (7) the equilibria are solutions of

$$\begin{aligned} &M_h[1 - M_h - (n-1)CM_hM_k^2] = 0\,,\\ &M_k[1 - M_k - CM_kM_h^2 - (n-2)CM_k^3] = 0\,,\\ &\beta_{hk} = CM_hM_k, \quad \beta_{kl} = CM_k^2\,, \end{aligned}$$

from which, using the result in subspace (5), we can write as

$$[(n-1)CM_k^3 + M_k - 1][(n-1)(n-2)C^2M_k^4 + (2n-3)CM_k^2 - CM_k + 1] = 0.$$

Then C^* exists,

$$C^* = C^*(n) = \frac{8}{\sqrt{(2n-3)^2[\gamma(n)]^2 + 108(n-1)(n-2)} - (2n-3)\gamma(n)}\,,$$

where $\gamma(n) = [32n(n-3) + 63]$, such that:

— for $C < C^*(n)$, one equilibrium $R = (r, r)$, where r is the only real root of $(n-1)CM_k^3 + M_k - 1 = 0$ exists, implying the presence of equilibrium $R = (r, r, \ldots, r)$ in the complete model;

— for $C > C^*(n)$, three equilibria

$$R = (r, r), \quad S = (b_1, s_1), \quad S^* = (b_2, s_2)\,,$$

exist for any n, where $b_i = 1/[(n-1)Cs_i^2+1]$, $i = 1, 2$, and s_1, s_2, with $s_1 < s_2$, are the only two real roots of $(n-1)(n-2)C^2M_k^4+(2n-3)CM_k^2-CM_k+1 = 0$, accordingly in the complete model, S, S^* imply the existence of $2n$ equilibria, beside R, of the form

$$\begin{aligned} S_h &= (s_1, \ldots, s_1, b_1, s_1, \ldots, s_1)\,,\\ S_h^* &= (s_2, \ldots, s_2, b_2, s_2, \ldots, s_2) \end{aligned} \qquad h = 1, \ldots, n\,,$$

i.e. having the M_i's all equal if $i \neq h$;

— at $C = C^*(n)$ S, S^* disappear by a saddle-node bifurcation and consequently the n S_h's and the n S_h^*'s disappear by n saddle-node bifurcations in the complete model.

It can be proven that no other equilibria can be found in (3). For example, in subspace (9) equilibria are solutions of

$$M_h\big[1 - M_h - (n_h-1)CM_h^3 - n_kCM_k^2M_h - n_lCM_l^2M_h\big] = 0\,, \tag{10}$$

$$M_k\big[1 - M_k - n_hCM_h^2M_k - (n_k-1)CM_k^3 - n_lCM_l^2M_k\big] = 0\,, \tag{11}$$

$$M_l\big[1 - M_l - n_hCM_h^2M_l - n_kCM_k^2M_l - (n_l-1)CM_l^3\big] = 0\,, \tag{12}$$

with $\beta_{hk} = M_h M_k$, $\beta_{kl} = M_k M_l$ and $\beta_{hl} = M_h M_l$. Subtracting (12) from (11) we have the only solution $M_l = M_k$. Subtracting (11) from (10) we have $M_k \neq M_h$ only if $n_h = 1$ and we obtain the points S_h and S_h^*, while $M_k = M_h$ if $n_h \geq 2$ and we have R.

Therefore, the reduced system (8) gives all the equilibria of the complete model. It is possible to prove that the reduced model also gives account for the stability properties of the equilibria in the complete model, which depend also on the adaptative rate T. The results, together with a detailed description of the equilibria and their dependence on the parameters, will be reported elsewhere, see [6].

We anticipate that the equilibria S_h^*'s, as the ones with one or more $M_i = 0$, are unstable for any value of the parameters, while the S_h's are stable for any value of C and n, if T is below a certain threshold at which a Hopf bifurcation takes place. Furthermore, for the equilibria S_h the activity level of node h is much higher than the ones of the other nodes.

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