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NUCLEAR CONTAMINATION IN A NATURALLY FRACTURED POROUS MEDIUM

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Abstract: Most of the natural reservoirs contain highly conductive fissures together with a large number of matrix blocks. A classical approach for their modeling uses the double porosity concept of Barenblatt et al [4] (see also Arbogast et al [3], Choquet [6]). The less permeable part of the rock contributes as global source terms for the transported solutes in the fracture. Using two-scale convergence and unfolding arguments, we derive rigorously a double porosity model for a miscible compressible flow of radionuclides.

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1. Introduction and Microscopic Model

We model the far-field repository by a domain $\Omega \subset \mathbb{R}^3$ with a periodic structure controlled by a parameter $\epsilon > 0$. The standard period is a unit cell Y consisting of a matrix block Y_m of external $\mathcal{C}^1$ boundary ∂Y_m completely surrounded by a fracture domain Y_f. The ϵ-reservoir consists of copies ϵY covering Ω. We set

$$\Omega_f^\epsilon = \Omega \cap \left\{\cup_{k\in\mathbb{Z}^3}\epsilon(Y_f+\xi)\right\}, \quad \Omega_m^\epsilon = \Omega \cap \left\{\cup_{k\in\mathbb{Z}^3}\epsilon(Y_m+\xi)\right\}.$$

The fracture-matrix interface is $\Gamma_{fm}^\epsilon = \partial\Omega_f^\epsilon \cap \partial\Omega_m^\epsilon$. The $\mathcal{C}^1$ boundary of Ω is Γ and ν is the exterior normal. The time interval of interest is $J = (0,T)$, $T > 0$. Our starting point is derived from the model in [7] for a radioactive transport problem:

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$$\oplus^\epsilon\,\partial_t\mathcal{P}^\epsilon+\mathrm{div}(\underline{\mathcal{U}}^\epsilon)=q,\quad \underline{\mathcal{U}}^\epsilon=-\frac{\mathcal{K}^\epsilon}{\mu(\mathcal{C}^\epsilon)}\nabla\mathcal{P}^\epsilon, \tag{1.1}$$

$$\oplus^\epsilon\,\partial_t\mathcal{C}^\epsilon+\underline{\mathcal{U}}^\epsilon\cdot\nabla\mathcal{C}^\epsilon-\mathrm{div}(\mathcal{E}_\epsilon(\underline{\mathcal{U}}^\epsilon)\nabla\mathcal{C}^\epsilon)=q(1-\mathcal{C}^\epsilon), \tag{1.2}$$

$$\mathcal{K}^\epsilon\nabla\mathcal{P}^\epsilon\cdot\nu=0,\quad \mathcal{E}_\epsilon(\underline{\mathcal{U}}^\epsilon)\nabla\mathcal{C}^\epsilon\cdot\nu=0\quad\text{on }\Gamma\times J, \tag{1.3}$$

$$\mathcal{P}^\epsilon(x,0)=P_0(x),\quad \mathcal{C}^\epsilon(x,0)=C_0(x)\quad\text{in }\Omega. \tag{1.4}$$

The pressure $\mathcal{P}^\epsilon=\chi_f^\epsilon P^\epsilon+\chi_m^\epsilon p^\epsilon$ and the concentration $\mathcal{C}^\epsilon=\chi_f^\epsilon C^\epsilon+\chi_m^\epsilon c^\epsilon$ are assumed continuous across the interface Γ_{fm}^ϵ, χ_i^ϵ being the characteristic function of Ω_i^ϵ. The Darcy velocity is $\underline{\mathcal{U}}^\epsilon$. Because of the structure of Ω, the porosity $(\Phi^\epsilon(x),\phi^\epsilon(x))=(\Phi(\frac{x}{\epsilon}),\phi(\frac{x}{\epsilon}))$ and the permeability $(K^\epsilon(x),k^\epsilon(x))=(K(\frac{x}{\epsilon}),k(\frac{x}{\epsilon}))$ are $(\epsilon Y_f,\epsilon Y_m)$-periodic and discontinuous across Γ_{fm}^ϵ. We set $\oplus^\epsilon=\chi_f^\epsilon\Phi^\epsilon+\chi_m^\epsilon\phi^\epsilon$, $\mathcal{K}^\epsilon=\chi_f^\epsilon K^\epsilon+\chi_m^\epsilon\epsilon^2k^\epsilon$, $\mathcal{E}_\epsilon(\underline{\mathcal{U}}^\epsilon)=\chi_f^\epsilon\mathcal{E}_c(x/\epsilon,U^\epsilon)+\chi_m^\epsilon\epsilon^2\mathcal{E}_c(x/\epsilon,\epsilon k^\epsilon\nabla p^\epsilon)$, where $\mathcal{E}_c(\underline{u})=\phi\big(D_mId+|\underline{u}|(\alpha_l\mathcal{D}(\underline{u})+\alpha_t(Id-\ \mathcal{D}(\underline{u})))\big)$, $\mathcal{D}(\underline{u})_{ij}=\underline{u}_i\underline{u}_j/|\underline{u}|^2$, $\alpha_l\geq\alpha_t\geq D_m>0$.

The assumptions are those used to state the existence of weak solutions for this type of problem in [7]: $q\in L^2(\Omega\times J)$, $0<\phi_-\leq\oplus(x)\leq\phi_-^{-1}$, $k_-|\xi|^2\leq\mathcal{K}(x)\xi\cdot\xi\leq k_-^{-1}|\xi|^2$, $k_->0$, a.e. in Ω, for all $\xi\in\mathbb{R}^3$; the viscosity $\mu\in W^{1,\infty}(0,1)$ is such that $0<\mu_-\leq\mu(x)\leq\mu_+\ \forall x\in(0,1)$; $P_0\in H^1(\Omega)$, and $C_0\in L^\infty(\Omega)$ satisfies $0\leq C_0(x)\leq 1$ a.e. in Ω.

The existence result [7] gives for the present piecewise structure:

Theorem 1.1. *Let $\epsilon>0$. There exists a weak solution $(\mathcal{P}^\epsilon,\mathcal{C}^\epsilon)$ of Pb.* (1.1)-(1.4) *which satisfies the following uniform estimates:*

(i) $\|\mathcal{P}^\epsilon\|_{L^\infty(J;L^2(\Omega))}+\|P^\epsilon\|_{L^2(J;H^1(\Omega_f^\epsilon))}+\|\epsilon p^\epsilon\|_{L^2(J;H^1(\Omega_m^\epsilon))}\leq C$. The time derivative $(\oplus^\epsilon\partial_t\mathcal{P}^\epsilon)$ is uniformly bounded in $L^2(J;(H^1(\Omega))')$.

(ii) $\|C^\epsilon\|_{L^2(J;H^1(\Omega_f^\epsilon))}+\|\epsilon c^\epsilon\|_{L^2(J;H^1(\Omega_m^\epsilon))}\leq C$, $0\leq\mathcal{C}^\epsilon(x,t)\leq 1$ a.e. in $\Omega\times J$, and $(\oplus^\epsilon\partial_t\mathcal{C}^\epsilon)$ is uniformly bounded in $L^2(J;(H^2(\Omega))')$.

2. The Macroscopic Model

2.1. The Limit Double Porosity Model

The aim of this paper is to derive rigorously the double porosity model described below: A macroscopic fracture system is driven by equations in $\Omega\times J$, similar to the microscopic ones:

$$\Phi^H\partial_tP+\mathrm{div}(\underline{U})=q-\int_{Y_m}\phi(y)\,\partial_tp\,dy,\quad \underline{U}=-\frac{K^H}{\mu(C)}\nabla P, \tag{2.1}$$

$$\Phi^H \partial_t C + \underline{U} \cdot \nabla C - \operatorname{div}(\mathcal{E}(\nabla P, \mu(C))\nabla C) = q(1 - |Y_f|C)$$
$$-q \int_{Y_m} c\, dy - \int_{Y_m} \phi(y)\, \partial_t c\, dy + \int_{Y_m} \frac{k(y)}{\mu(c)} \nabla_y p \cdot \nabla_y c\, dy, \qquad (2.2)$$
$$K^H \nabla P \cdot \nu = 0,\ \mathcal{E}\nabla C \cdot \nu = 0 \text{ on } \Gamma \times J, \qquad (2.3)$$
$$P(x,0) = P_0(x),\ C(x,0) = C_0(x) \quad \text{in } \Omega \times \{0\}. \qquad (2.4)$$

The effective porosity, permeability and diffusion tensor are

$$\Phi^H(x) = \int_{Y_f} \Phi(y) dy, \qquad (2.5)$$

$$K^H_{ij}(x) = \int_{Y_f} K(y)(\nabla_y v_i(y) + e_i) \cdot (\nabla_y v_j(y) + e_j) dy, \qquad (2.6)$$

$$\mathcal{E}_{ij}(x) = \int_{Y_f} \mathcal{E}_f(\nabla_y w_i(x,y) + e_i) \cdot (\nabla_y w_j(x,y) + e_j) dy, \qquad (2.7)$$

where $\mathcal{E}_f = \mathcal{E}_c\Big(\frac{K_f}{\mu(C)}\nabla P\Big)$ and $K_{f_{ij}} = K(\nabla_y v_i + e_i) \cdot (\nabla_y v_j + e_j)$. The functions $(v_i)_{1 \le i \le 3}$ and $(w_i)_{1 \le i \le 3}$ are Y-periodic solutions of the cell problems (2.8) and (2.9) below.

$$\begin{aligned} -\operatorname{div}_y(K(y)(\nabla_y v_i(y) + e_i)) &= 0 \text{ in } Y_f, \\ K(y)(\nabla_y v_i(y) + e_i) \cdot \nu_y &= 0 \text{ on } \Gamma_{fm}, \end{aligned} \qquad (2.8)$$

$$\begin{aligned} -\operatorname{div}_y(\mathcal{E}_f(x,y)(\nabla_y w_i(x,y) + e_i)) &= 0 \text{ in } Y_f, \\ \mathcal{E}_f(x,y)(\nabla_y w_i(x,y) + e_i) \cdot \nu_y &= 0 \text{ on } \Gamma_{fm}. \end{aligned} \qquad (2.9)$$

Fo each $x \in \Omega$ a matrix block is driven by equations in $\{x\} \times Y_m \times J$ which give the additional source terms:

$$\phi(y)\, \partial_t p + \operatorname{div}_y(\underline{u}) = q, \quad \underline{u} = -(1/\mu(c)) k(y) \nabla_y p, \qquad (2.10)$$
$$\phi(y)\, \partial_t c + \underline{u} \cdot \nabla_y c - \operatorname{div}_y(\mathcal{E}_c(y, \underline{u}) \nabla_y c) = q(1-c), \qquad (2.11)$$
$$p = P,\ c = C \text{ on } \partial Y_m, \qquad (2.12)$$
$$p(x,y,0) = P_0(x),\ c(x,y,0) = C_0(x) \quad \text{in } Y_m \times \{0\}. \qquad (2.13)$$

Theorem 2.1. *As $\epsilon \to 0$, the microscopic model* (1.1)–(1.4) *converges to the double porosity macroscopic model* (2.1)–(2.13).

2.2. The Homogenization Process

Theorem 1.1 gives sufficient estimates to state two-scale convergence results (see Allaire [2]). Following [1], we also extend the functions P^ϵ and C^ϵ to the whole domain Ω with a linear and continuous extension operator $\Pi^\epsilon : H^1(\Omega^\epsilon_f) \to H^1_{loc}(\Omega)$. We thus ensure that, up to subsequences, we have the following convergences

$$\Pi^\epsilon P^\epsilon \rightharpoonup P \text{ weakly in } L^2(J;H^1(\Omega)),\ \nabla(\Pi^\epsilon P^\epsilon) \overset{2}{\rightharpoonup} \nabla P + \nabla_y P^1,$$

$$\Pi^\epsilon C^\epsilon \rightharpoonup C \text{ weakly in } L^2(J;H^1(\Omega)),\ \nabla(\Pi^\epsilon C^\epsilon) \overset{2}{\rightharpoonup} \nabla C + \nabla_y C^1,$$

$$\mathcal{P}^\epsilon \overset{2}{\rightharpoonup} p, \quad \epsilon\nabla\mathcal{P}^\epsilon \overset{2}{\rightharpoonup} \nabla_y p, \quad \mathcal{C}^\epsilon \overset{2}{\rightharpoonup} c, \quad \epsilon\nabla\mathcal{C}^\epsilon \overset{2}{\rightharpoonup} \nabla_y c,$$

with $P \in L^2(J;H^1(\Omega))$, $P^1 \in L^2(\Omega\times J;H^1_{per}(Y))$, $p \in L^2(\Omega\times J;H^1_{per}(Y))$, $C \in L^2(J;H^1(\Omega))$, $C^1 \in L^2(\Omega\times J;H^1_{per}(Y))$ and $c \in L^2(\Omega\times J;H^1_{per}(Y))$. Using the uniform estimate on $\oplus\partial_t\mathcal{C}^\epsilon$ and a compactness argument of Aubin's type, we moreover assert that $\Pi^\epsilon C^\epsilon \to C$ strongly in $L^2(\Omega\times J)$. The functions $\oplus^\epsilon$ and $\mathcal{K}^\epsilon$ are such that $\oplus^\epsilon \overset{2}{\rightharpoonup} \chi_f(y)\phi_f(y) + \chi_m(y)\phi(y)$, $\mathcal{K}^\epsilon \overset{2}{\rightharpoonup} \chi_f(y)k_f(y)$ $+\chi_m(y)k(y)$.

We have the first tools to study the behavior of the microscopic system as ϵ tends to zero. We begin with the pressure problem. Classical arguments (see [2]) give the following incomplete result:

$$\Phi^H\,\partial_t P - \mathrm{div}\big((1/\mu(C))K^H\nabla P\big) = q - \int_{Y_m}\phi\,\partial_t p\,dy \quad \text{in } \Omega\times J,$$

$$\phi\,\partial_t p + \mathrm{div}_y\underline{u} = q \quad \text{in } \Omega\times Y_m\times J,$$

$$K^H\nabla P\cdot\nu = 0 \text{ on } \partial\Omega\times J,\ P(x,0) = p(x,y,0) = P_0(x) \text{ in } \Omega\times Y_m.$$

We now have to determine the nonexplicit limit $\underline{u}$. We use unfolding arguments to reach directly the scale of the standard cell (see Arbogast et al [3] and Cioranescu et al [8]).

2.3. Introduction of an Appropriate Dilation Operator

For each $\epsilon > 0$, we define a dilation operator $\widetilde{\cdot}$ mapping measurable functions on $\Omega_m^\epsilon\times J$ to measurable functions on $\Omega\times Y_m\times J$ by

$$\widetilde{v}(x,y,t) = v(\epsilon k_x + \epsilon y, t) \quad \text{for } y\in Y_m,\ (x,t)\in\Omega\times J,$$

where ϵk_x is the lattice translation point of the ϵ-cell domain containing x. We extend this operator from Y_m to $\cup_k(Y_m+k)$ periodically. Any function $v\in L^2(J;H^1(\Omega_m^\epsilon))$ satisfies

$$\|\widetilde{v}\|_{L^2(\Omega_m^\epsilon\times J\times Y_m)} = \|v\|_{L^2(\Omega\times J)}, \quad \nabla_y\widetilde{v} = \epsilon\,\widetilde{\nabla_x v} \text{ a.e. in } \Omega\times J\times Y_m\,,$$

$$\|\nabla_y\widetilde{v}\|_{(L^2(\Omega\times J\times Y_m))^3} = \epsilon\big\|\widetilde{\nabla_x v}\big\|_{(L^2(\Omega_m^\epsilon\times J))^3}.$$

The two-scale convergence is linked with the weak convergence of dilated sequences (Bourgeat et al [5]): if $\widetilde{v^\epsilon} \rightharpoonup v^o$ weakly in $L^2(\Omega\times J;L^2_{per}(Y_m))$ and

$\chi_m^\epsilon v^\epsilon \overset{2}{\rightharpoonup} v$, then $v^o = v$ a.e. in $\Omega \times J \times Y_m$. We thus only have to determine the limit behavior of the dilated solutions $\widetilde{c}^\epsilon$ and $\widetilde{p}^\epsilon$. They are weak solutions in $\Omega \times Y_m \times J$ of

$$\phi(y)\,\partial_t \widetilde{p}^\epsilon + \mathrm{div}_y \widetilde{u}^\epsilon = \widetilde{q}, \quad \widetilde{u}^\epsilon = -(1/\mu(\widetilde{c}^\epsilon))k(y)\nabla_y \widetilde{p}^\epsilon, \tag{2.14}$$

$$\phi(y)\,\partial_t \widetilde{c}^\epsilon + \widetilde{u}^\epsilon \cdot \nabla_y \widetilde{c}^\epsilon - \mathrm{div}_y(\mathcal{E}(y, \widetilde{u}^\epsilon)\nabla_y \widetilde{c}^\epsilon) = \widetilde{q}\,(1 - \widetilde{c}^\epsilon), \tag{2.15}$$

$$\widetilde{p}^\epsilon = \widetilde{P^\epsilon} \quad \text{and} \quad \widetilde{c}^\epsilon = \widetilde{C^\epsilon} \quad \text{in } H^{1/2}(\partial Y_m) \text{ for } (x,t) \in \Omega \times J, \tag{2.16}$$

$$\widetilde{p}^\epsilon(x,y,0) = \widetilde{P_0}(x,y), \quad \widetilde{c}^\epsilon(x,y,0) = \widetilde{C_0}(x,y) \quad \text{in } \Omega \times Y_m. \tag{2.17}$$

Using the basic properties of the dilation operator and the estimates of Theorem 1.1, we claim that, for subsequences,

$$\widetilde{p}^\epsilon \rightharpoonup p,\ \widetilde{c}^\epsilon \rightharpoonup c \quad \text{weakly in } L^2(\Omega \times J \times Y_m),$$

$$\nabla_y \widetilde{p}^\epsilon \rightharpoonup \nabla_y p,\ \nabla_y \widetilde{c}^\epsilon \rightharpoonup \nabla_y c \quad \text{weakly in } (L^2(\Omega \times J \times Y_m))^3.$$

Note that the dilation operator has killed the penalizing ϵ in some gradients estimates. We clearly also can get some uniform estimates for $\phi(y)\partial_t \widetilde{c}^\epsilon$. But this is not sufficient to get compactness results because we have no estimate for $\nabla_x \widetilde{c}^\epsilon$. Let then choose $k \in \mathbb{Z}^3$. For $f^\epsilon = p^\epsilon$ or c^ϵ, we define $\widetilde{f}_k^\epsilon$ in $Y_m \times J$ by

$$\widetilde{f}_k^\epsilon(y,t) = \begin{cases} \widetilde{f}^\epsilon(x,y,t)_{/x \in \epsilon(Y_m + k)} & \text{if } k \text{ is such that } \epsilon(Y_m + k) \cap \Omega \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Now x is a parameter. We have enough convergence results (analogous to those obtained for $\Pi^\epsilon P^\epsilon$ and $\Pi^\epsilon C^\epsilon$) to find the equations satisfied by the limit (p_k, c_k) of $(\widetilde{p}_k^\epsilon, \widetilde{c}_k^\epsilon)$. We get firstly

$$\phi(y)\,\partial_t p_k - \mathrm{div}_y((1/\mu(c_k))k(y)\nabla_y p_k) = q \quad \text{in } Y_m \times J,\ \forall k \in \mathbb{Z}^3.$$

By a density argument [6], we conclude that p is solution of (2.10). Moreover, comparing the variational formulations associated to (1.1) in the matrix part of the domain and to (2.10), one shows that

$$\lim_{\epsilon \to 0} \| \frac{\epsilon}{\mu(c^\epsilon)^{1/2}} \nabla p^\epsilon \|_{(L^2(\Omega_m^\epsilon \times J))^3} = \| \frac{1}{\mu(c)^{1/2}} \nabla_y p \|_{(L^2(\Omega \times J \times Y_m))^3}. \tag{2.18}$$

This latter result let us pass to the limit in the matrix part of equation (1.2) in spite of the nonlinearities depending on the Darcy velocity $\underline{u}^\epsilon$. We get (2.11). It remains to justify (2.2), that is to pass to the limit in the fractured part of equation (1.2).

2.4. Corrector for the Flux Function

We derive a corrector for the Darcy velocity in the fracture. Comparing the variational formulation of equation (1.1) with the one of its two-scale limit, we infer from (2.18) and $\Pi^\epsilon C^\epsilon \to C$ a.e. in $\Omega \times J$ that

$$\left\| \chi_f^\epsilon \frac{\mathcal{K}^\epsilon}{\mu(\mathcal{C}^\epsilon)} \nabla \mathcal{P}^\epsilon \right\|_{(L^2(\Omega\times J))^3} \to \left\| \chi_f(y) \frac{K(y)}{\mu(C)} (\nabla P + \nabla_y P^1) \right\|_{(L^2(\Omega\times J\times Y))^3}.$$

This relation is sufficient to claim the following result.

Proposition 2.2. *Let us define* $\underline{\mathcal{U}}_o$ *by*

$$\underline{\mathcal{U}}_o(x,y,t) = -\chi_f(y) \frac{K(y)}{\mu(C(x,t))} (\nabla P(x,t) + \nabla_y P^1(x,y,t)).$$

Assume that $\lim_{\epsilon\to 0} \int_{\Omega\times J} |\underline{\mathcal{U}}_0(x,x/\epsilon,t)|^2 \le \int_{\Omega\times J}\int_Y |\underline{\mathcal{U}}_0(x,y,t)|^2$. *The function* $\underline{\mathcal{U}}_0(x,x/\epsilon,t)$ *is a corrector for the flux function:*

$$\lim_{\epsilon\to 0} \left\| -\chi_f^\epsilon \frac{\mathcal{K}^\epsilon}{\mu(\mathcal{C}^\epsilon)} \nabla \mathcal{P}^\epsilon - \underline{\mathcal{U}}_0(x, \frac{x}{\epsilon}, t) \right\|_{(L^2(\Omega\times J))^3} = 0.$$

We then can substitute $\underline{\mathcal{U}}_0(x,x/\epsilon,t)$ to $\chi_f^\epsilon (K^\epsilon/\mu(\mathcal{C}^\epsilon))\nabla \mathcal{P}^\epsilon$ to pass to the limit in the concentration equation. We get (2.2). This completes the proof of Theorem 2.1.

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