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# THE VELOCITY TRACKING PROBLEM FOR MHD FLOWS WITH DISTRIBUTED MAGNETIC FIELD CONTROLS

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**Abstract:** We consider the mathematical formulation and the analysis of an optimal control problem associated with the tracking of the velocity of a viscous, incompressible, electrically conducting fluid in a bounded two-dimensional domain through the adjustment of distributed magnetic field controls.

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## 1. Introduction

Control methods for fluid dynamics have attracted substantial interest in recent years due to their wide range of applications in engineering and science. In the literature, examples are found related to control of plasma current in tokamak, combustion, chemical reacting flows, design problems, reduction of turbulence, drag reduction; see, e.g., Ambrosino et al [2], Glowinski et al [4], Gunzburger [5], Jameson [6], McManus et al [8] and Turner [11].

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In this paper we study an optimal control problem for viscous, incompressible, electrically conducting fluid, where the control applied is a distributed electric current, and the goal is to match the controlled velocity to some given field. The controls and the states are constrained to satisfy a coupled system consisting of the Navier-Stokes and Maxwell's equations.

The mathematical description of the control problem proceeds as follows. Let $\Omega$ be a bounded, connected open domain of class $C^{1,1}$ or a convex polyhedron. Let $u, p, B$ denote the velocity, pressure and magnetic fields, respectively. We denote $\mathrm{curl}\psi$ the magnetic field control and $f$ the given body force. Also let $\nu$ denote the outward unit normal vector on $\partial\Omega$. For given $T$, we define the following cost functional:

$$\mathcal{J}(u, B, \mathrm{curl}\psi) = \int_0^T \int_\Omega \left( \frac{\alpha}{2}|u - u_d|^2 + \frac{\beta S}{2}|\mathrm{curl}\psi|^2 \right) dxdt \tag{1}$$

where $u_d$ is some desired velocity field.

Our goal is to minimize (1) subject to the MHD equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \frac{1}{\mathrm{Re}}\Delta u - S\mathrm{curl}B \times B + \nabla p = f \quad \text{in } \Omega \times (0,T), \tag{2a}$$

$$\frac{\partial B}{\partial t} + \frac{1}{\mathrm{Re_m}}\mathrm{curl}(\mathrm{curl}B) - \mathrm{curl}(u \times B) = \mathrm{curl}\psi \quad \text{in } \Omega \times (0,T), \tag{2b}$$

$$\mathrm{div}u = 0, \quad \mathrm{div}B = 0, \quad \text{in } \Omega \times (0,T), \tag{2c}$$

$$u = 0, \quad B \cdot \nu = 0 \quad \text{and } \mathrm{curl}B \times \nu = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2d}$$

$$u(\cdot, 0) = u_0(\cdot), \quad B(\cdot, 0) = B_0(\cdot) \quad \text{in } \Omega. \tag{2e}$$

Here $\mathrm{Re}, \mathrm{Re_m}$, and $S$ are nondimensional constants that characterize the flow: the Reynolds number, the magnetic Reynolds number and the coupling number, respectively , Sermange et al [9]).

## 2. The Optimal Control Problem

We use the standard notations (see e.g. Adams [1] and Girault et al [3]) for the Sobolev spaces $H^m(\Omega)$. We recall $L_0^2(\Omega) = \left\{p \in L^2(\Omega) : \int_\Omega q d\Omega = 0\right\}$, equipped with $L^2(\Omega)$ inner product, and the solenoidal spaces:

$$H = \{u \in L^2(\Omega);\ \mathrm{div}\ u = 0 \text{ and } u \cdot \nu = 0\}^2, \quad V = H \cap (H_0^1(\Omega) \times H_\nu^1(\Omega)).$$

The boundary condition $\mathrm{curl}B \times \nu = 0$ is enforced weakly through the variational equation and is not imposed on the definition of $V$. In order to define a weak form of the MHD equations we introduce the following linear forms. For

$(u_i, B_i) \in H^1_0(\Omega) \times H^1_\nu(\Omega)$, $q \in L^2_0(\Omega)$

$$a((u_1, B_1),(u_2, B_2)) = \frac{1}{\mathrm{Re}}\int_\Omega \nabla u_1 : \nabla u_2 dx + \frac{S}{\mathrm{Re_m}}\int_\Omega \mathrm{curl} B_1 \mathrm{curl} B_2 dx, \tag{3a}$$

$$b((u_1, B_1), q) = -\int_\Omega q\nabla \cdot u_1 dx, \tag{3b}$$

$$c((u_1, B_1),(u_2, B_2),(u_3, B_3)) = \int_\Omega (u_1 \cdot \nabla)u_2 \cdot u_3 dx \tag{3c}$$

$$-S\int_\Omega \mathrm{curl} B_2 \times B_1 \cdot u_3 dx - S\int_\Omega \left[(u_1 \cdot \nabla)B_2 \cdot B_3 - (B_1 \cdot \nabla)u_2 \cdot B_3\right] dx.$$

We define the diagonal matrix $\mathcal{S} \in \mathcal{M}_4(\mathbb{R}^2)$ by $m_{ii} = 1$ if $1 \le i \le 2, m_{ii} = S$ if $3 \le i \le 4$. A weak formulation of the MHD problem is defined as follows: seek $(u, B, p) \in L^2(0, T; H^1_0(\Omega)) \times L^2(0, T; H^1_\nu(\Omega)) \times L^2(0, T; L^2_0(\Omega))$ satisfying

$$\langle \frac{d}{dt}(u,B), \mathcal{S}(v,C)\rangle + a((u,B),(v,C)) + c((u,B),(u,B),(v,C)) + b((v,C),p) \tag{4a}$$

$$= \langle (f, \mathrm{curl}\psi), \mathcal{S}(v, C)\rangle \quad \forall (v, C) \in H^1_0(\Omega) \times H^1_\nu(\Omega),$$

$$b((u, B), q) = 0 \quad \forall q \in L^2_0(\Omega), \tag{4b}$$

$$(u, B)(0) = (u_0, B_0). \tag{4c}$$

The set of all possible target velocities $L^\infty(0, T; L^2(\Omega))$ is denoted $\mathcal{U}_{ad}$. There are no particular requirements on the target velocities $u_d$ other than the fact that the cost functional must be bounded. The target needs not to be a solution to the MHD system. In particular, non solenoidal fields that satisfy boundary and initial conditions different from those in (2d) and (2e) can be used as desired target velocities.

The set of all admissible solutions is defined by

$$\mathcal{A}_{ad} = \{(u, B, \mathrm{curl}\psi) \in L^2(0, T; H^1_0(\Omega)) \times L^2(0, T; H^1_\nu(\Omega)) \times L^2(0, T; L^2(\Omega)),$$
$$\mathcal{J}(u, B, \mathrm{curl}\psi) < \infty, \text{and } \exists\ p \in L^2(0, T; L^2_0(\Omega)) \text{ s.t. (4a)-(4c)}\}.$$

With this notation, the formulation of the optimal control problem is:

*given* $(u_0, B_0) \in V, u_d \in \mathcal{U}_{ad}$, *find* $(\hat{u}, \hat{B}, \mathrm{curl}\hat{\psi}) \in \mathcal{A}_{ad}$ *s.t.* (1) *is minimized.*

We recall that if $\partial\Omega$ is Lipschitz continuous, $(f, \mathrm{curl}\psi) \in L^2(0, T; V')$, and $(u_0, B_0) \in H$, then the unique solution to (4a)-(4c) satisfies $(u, B) \in C([0, T]; H) \cap L^2(0, T; V)$ and $(u_t, B_t) \in L^2(0, T; V')$. If $\partial\Omega$ is $C^2$, $(u_0, B_0) \in V$ and $(f, \mathrm{curl}\psi) \in L^2(0, T; H)$, then $(u, B) \in C([0, T], V) \cap L^2(0, T; H^2(\Omega) \cap V)$.

**Theorem 1.** *Given* $T > 0$, $(u_0, B_0) \in V$, *and* $u_d \in \mathcal{U}_{ad}$, *then there exists a solution* curl$\hat{\psi} \in L^2(0, T; L^2(\Omega))$ *and* $(\hat{u}, \hat{B}) \in C([0, T]; H) \cap L^2(0, T; V)$ *of the optimal control problem.*

The optimal solution must satisfy the first-order necessary condition associated with the optimal control problem. By studying the case in which the Gâteaux derivative of the cost functional vanishes, we get a possible candidate solution for the optimal control (see [10]).

**Theorem 2.** *The mapping* $\mathrm{curl}\widetilde{\psi} \mapsto ((u,B))(\mathrm{curl}\widetilde{\psi})$ *from* $L^2(0,T;L^2(\Omega))$ *to* $L^2(0,T;V)$ *has a Gâteaux derivative* $\Big(d(u,B)/d(\mathrm{curl}\widetilde{\psi})\Big)\cdot \mathrm{curl}\psi$ *for every* $\mathrm{curl}\psi \in L^2(0,T;L^2(\Omega)$. *Moreover,* $((\check{u},\check{B}))(\mathrm{curl}\psi) = \Big(d(u,B)/d(\mathrm{curl}\widetilde{\psi})\Big)\cdot\mathrm{curl}\psi$ *is the solution of the linear problem*

$$\begin{aligned}\langle\frac{d}{dt}(\check{u},\check{B}),\mathcal{S}(v,C)\rangle + a((\check{u},\check{B}),(v,C)) + c((\check{u},\check{B}),(u,B),(v,C))\\ + c((u,B),(\check{u},\check{B}),(v,C)) + b((v,C),\check{r}) = \langle(0,\mathrm{curl}\psi),\mathcal{S}(v,C)\rangle\\ \forall (v,C)\in H_0^1(\Omega)\times H_\nu^1(\Omega),\end{aligned} \tag{5a}$$

$$b((\check{u},\check{B}),q) = 0\,,\quad \forall q\in L_0^2(\Omega), \tag{5b}$$

$$(\check{u},\check{B})(0) = (0,0), \tag{5c}$$

*where* $\check{r}\in L^2(0,T;L_0^2(\Omega))$, *and* $(\check{u},\check{B})\in L^\infty(0,T;H)\cap L^2(0,T;V)$.

Since the Gâteaux derivative of the functional $\mathcal{J}$ should vanish at the optimal solution, the optimal control $\mathrm{curl}\hat{\psi}$ must be proportional to the solution of the linear adjoint system.

**Theorem 3.** *Let* $(\hat{u},\hat{B},\hat{p},\mathrm{curl}\hat{\psi})$ *be a solution of the optimal control problem and* $(w,D,r)\in L^\infty(0,T;H)\cap L^2(0,T;V)$ *be a solution of the adjoint problem*

$$\begin{aligned}-\langle\frac{d}{dt}(w,D),\mathcal{S}(v,C)\rangle + a((w,D),(v,C)) + c((v,C),(\hat{u},\hat{B}),(w,D))\\ + c((\hat{u},\hat{B}),(v,C),(w,D)) + b((v,C),r) = \langle(\alpha(\hat{u}-u_d),0),\mathcal{S}(v,C)\rangle\end{aligned} \tag{6a}$$

$$b((w,D),q) = 0 \quad \forall q\in L_0^2(\Omega), \tag{6b}$$

$$(w,D)(T) = (0,0). \tag{6c}$$

*Then*

$$\beta\,\mathrm{curl}\hat{\psi} = -D. \tag{6d}$$

Hence the solutions of the optimal control problem are among the solutions of a optimality system, consisting in the forward MHD equation (4a)-(4c), the backward in time adjoint equation (6a)-(6c) and the optimality condition (6d). To solve this system, we need to discretize the problem in time and space.

## 3. Numerical Approximations

Let $\sigma_N = \{t_n\}_{n=0}^N$ be a partition of $[0,T]$ into equal intervals $\Delta t = T/N$ with $t_0 = 0$ and $t_N = T$. For each fixed $\Delta t$ and for every quantity $q(t,x)$ we associate the corresponding set $\{q^{(n)}(x)\}_{n=0}^N$ and a continuous piecewise linear function $q^N = q^N(t,x)$ such that $q^N(t_n,x) = q^{(n)}(x)$ for all $n = 0,\dots,N$. We will denote by $\mathbf{q}$ the vector $(q^{(1)},\dots,q^{(N)})$ of functions belonging to $\mathbf{X} = X^N$ and defined discretely with respect to time.

**Theorem 4.** *Given $T$, $\Delta t = T/N$, $(u_0,B_0) \in V$, and $u_d \in \mathcal{U}_{ad}$, there exists at least one solution $(\mathbf{u},\mathbf{B},\mathbf{p},\mathrm{curl}\boldsymbol{\psi}) \in \mathbf{H}_0^1(\Omega)\times\mathbf{H}_\nu^1(\Omega)\times\mathbf{L}_0^2(\Omega)\times\mathbf{L}^2(\Omega)$ of the semidiscrete-in-time optimal control problem.*

Now we state the convergence of the semidiscrete optimal control problem.

**Theorem 5.** *Given $\Delta t = T/N$, $u_d \in \mathcal{U}_{ad}$ and $(u_0,B_0) \in V$. For $\Delta t \to 0$, the solution $\{(u^{(n)},B^{(n)},p^{(n)},\mathrm{curl}\psi^{(n)})\}_{n=1}^N$ of the semidiscrete-in-time optimal control problem tends to the solution $(\hat{u},\hat{B},\hat{p},\mathrm{curl}\hat{\psi})$ of the corresponding continuous optimal control problem.*

We consider conforming finite element approximations. Let $X^h \subset H^1(\Omega)$, $Y^h(\Omega) \subset H^1(\Omega)$, and $S^h(\Omega) \subset L^2(\Omega)$ be three families of finite dimensional spaces parametrized by $h$. We define $X_0^h = X^h \cap H_0^1(\Omega)$, $Y_\nu^h = Y^h \cap H_\nu^1$ with corresponding norms induced by the norm on $H^1(\Omega)$, $S_0^h = S^h \cap L_0^2(\Omega)$ , $Z^h = \left\{u^h \in X_0^h; \int_\Omega q^h \nabla\cdot u^h dx = 0 \ \forall q \in S_0^h\right\}$, and the product spaces $V^h = X^h \times Y^h$, $V_{0\nu}^h = X_0^h \times Y_\nu^h$.

In order to have stable and accurate approximations, we assume that on $X_0^h$ and $S_0^h$ have been chosen so that the following inf-sup condition is satisfied on the finite-dimensional spaces, i.e., there exists a constant $\gamma^h > 0$ such that

$$\inf_{p_h \in S_0^h} \sup_{(u_h,B_h)\in V_{0\nu}^h} \frac{b((u_h,B_h),p_h)}{\|(u_h,B_h)\|_1 \|p_h\|_0} \geq \gamma^h.$$

Given $\Delta = T/N$, $(\boldsymbol{f}_h,\mathrm{curl}\boldsymbol{\psi}_h) \in \mathbf{L}^2(\Omega)\times\mathbf{L}^2(\Omega)$ and $(u_0,B_0) \in V$, $(\mathbf{u}_h,\mathbf{B}_h)$ is called a generalized solution of the fully discrete time-space approximation of the MHD equations if $(u_h^{(n)},B_h^{(n)}) \in V_{0\nu}^h$ and $p_h^{(n)} \in S_0^h$ and $(u_h^{(n)},B_h^{(n)},p_h^{(n)})$ satisfy

$$\begin{aligned} &\frac{1}{\Delta t}((u_h^{(n)}-u_h^{(n-1)},B_h^{(n)}-B_h^{(n-1)}),\mathcal{S}(v_h,C_h)) + a((u_h^{(n)},B_h^{(n)}),(v_h,C_h)) \qquad (7a)\\ &\qquad + \widetilde{c}((u_h^{(n)},B_h^{(n)}),(u_h^{(n)},B_h^{(n)}),(v_h,C_h)) + b((v_h,C_h),p_h^{(n)})\\ &\qquad = \langle (f_h^{(n)},\mathrm{curl}\psi_h^{(n)}),\mathcal{S}(v_h,C_h)\rangle\,, \quad \forall (v_h,C_h) \in V_{0\nu}^h,\\ &b((u_h^{(n)},B_h^{(n)}),q_h) = 0\,, \quad \forall q_h \in S_0^h, \qquad (7b)\end{aligned}$$

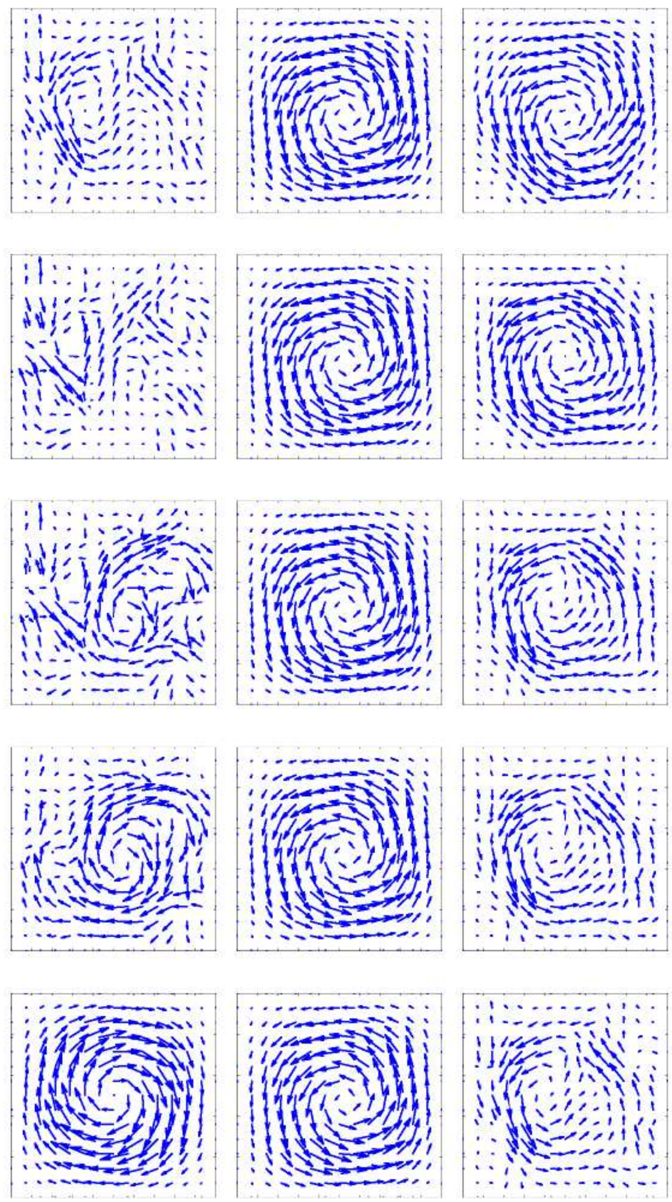

Figure 1: Controlled velocity $u$ at $t$ =0, .025, .03, .035, .04, .045, .05, .06, .1, .9

for $n=1,\ldots,N$, with initial data $(u_h^{(0)},B_h^{(0)})=\Pi^h(u_0(x),B_0(x))$, where $\Pi^h$ denotes the projection of $(u_0(x),B_0(x))$ onto $V_{0\nu}^h$.

The discrete functional used in the optimal control problem is given by

$$\mathcal{J}_h^N(\mathbf{u}_h,\mathbf{B}_h,\mathrm{curl}\boldsymbol{\psi}_h)=\frac{\alpha\Delta t}{2}\sum_{n=1}^{N}\|u_h^{(n)}-u_{hd}^{(n)}\|_0^2+\frac{\beta S\Delta t}{2}\sum_{n=1}^{N}\|\mathrm{curl}\psi_h^{(n)}\|_0^2.$$

The formulation of the fully discrete optimal control problem is *given* $\Delta t=T/N$, $(u_0,B_0)\in V$ *and* $u_d\in\mathcal{U}_{ad}$, *find* $(\hat{\mathbf{u}}_h,\hat{\mathbf{B}}_h,\hat{\mathbf{p}}_h,\mathrm{curl}\hat{\boldsymbol{\psi}}_h)\in V_{0\nu}^h\times S_0^h\times S^h$ *such that* (7*a*) – (7*b*) *are satisfied and the cost functional is minimized.*

**Theorem 6.** *Let* $(\hat{\mathbf{u}}_h,\hat{\mathbf{B}}_h,\hat{\mathbf{p}}_h,\mathrm{curl}\hat{\boldsymbol{\psi}}_h)$ *denote a solution of the fully discrete optimal control problem. Then we have*

$$\beta\mathrm{curl}\hat{\psi}_h^{(n)}=-D_h^{(n-1)},\quad \forall n=1,\ldots,N, \tag{8}$$

*where the functions* $(w_h^{(n)},D_h^{(n)})\in V_{0\nu}^h$, $r_h^{(n)}\in S_0^h$, $n=0,\ldots,N$, *satisfy*

$$\begin{aligned}
&-\tfrac{1}{\Delta t}\Big((w_h^{(n)},D_h^{(n)})-(w_h^{(n-1)},D_h^{(n-1)}),\mathcal{S}(v_h,C_h)\Big) \qquad (9a)\\
&\quad+a((v_h,C_h),(w_h^{(n-1)},D_h^{(n-1)}))+\widetilde{c}((v_h,C_h),(\hat{u}_h^{(n)},\hat{B}_h^{(n)}),(w_h^{(n-1)},D_h^{(n-1)}))\\
&\quad+\widetilde{c}((\hat{u}_h^{(n)},\hat{B}_h^{(n)}),(v_h,C_h),(w_h^{(n-1)},D_h^{(n-1)}))+b((v_h,C_h),r_h^{(n-1)})\\
&\quad=\Big((\alpha(\hat{u}_h^{(n)}-u_d^{(n)}),0),\mathcal{S}(v_h,C_h)\Big),\quad \forall(v_h,C_h)\in V_{0\nu}^h,
\end{aligned}$$

$$b((w_h^{(n-1)},D_h^{(n-1)}),q_h)=0,\quad \forall q_h\in S_0^h(\Omega), \tag{9b}$$

*for* $n=1,\ldots,N$ *along with the terminal condition* $(w_h^{(N)},D_h^{(N)})=(0,0)$.

We consider the unit square domain $(0,1)\times(0,1)\subset\mathbb{R}^2$, $T=1$. The finite element spaces are chosen to be continuous piecewise biquadratic polynomials for the velocity and magnetic field, and piecewise bilinear polynomials for the pressure, i.e., the Taylor-Hood finite element pair. A simple stationary target velocity $u_d(x,y)=(u_d^1(x,y),u_d^2(x,y))$ is chosed; e.g., $u_d^1=10\frac{d}{dy}\big(\phi(0.4,x)\phi(0.4,y)\big)$, $u_d^2=-10\frac{d}{dx}\big(\phi(0.4,x)\phi(0.4,y)\big)$, where $\phi(t,z)=(1-z)^2(1-\cos(2\pi tz))$. For the initial velocity we choose $u_0(x,y)=-u_d(x,y)$, $v_0(x,y)=-v_d(x,y)$ so that the initial flow rotates in an opposite sense from the target flow. The evolution of the velocity is given in Figure 1. The controlled fluid is showed on the top and bottom lines, and the desired steady flow is on the middle line.

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