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MEROMORPHIC CONTINUATION OF THE SCATTERING MATRIX

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Abstract: Long range quantum mechanical scattering in $\mathbb{R}^n$ in the presence of a constant electric field of strength $F > 0$ is discussed. The scattering matrix, as a function of energy, is shown to have a meromorphic continuation to the entire complex plane as a bounded operator on $L(\mathbb{R}^{n-1})$. This result differs from the comparable result in the Schrödinger case ($F = 0$).

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1. Introduction

The scattering matrix $S(\lambda)$ is an operator-valued function of energy λ. Our main result here is that $S(\lambda)$ extends to a meromorphic function on the entire complex plane $\mathbb{C} \ni \lambda$. This is of interest because the poles of the extension are resonances of the system which can be used to understand the scattering process.

Meromorphic continuation of the scattering matrix has been considered elsewhere [2, 5, 9, 11]; however this result is novel in the following physical setting. We consider potential scattering of a single charged quantum particle in $\mathbb{R}^n$ by a long range potential V. We further suppose that the scattering occurs in the presence of a constant electric field of strength F acting in the $\mathbf{e}_1$ direction of $\mathbb{R}^n$. The results here are new only in the case $F > 0$ (the "Stark" case) but we shall compare them to known results about the Schrödinger case ($F = 0$) because there are similarities but also some differences.

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We introduce some notation: The Hamiltonians in this context are

$$H_0 = -\Delta + Fx_1 \quad \text{and} \quad H = -\Delta + Fx_1 + V(x)\,,$$

where Δ is the Laplacian $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$ and H_0 is the "free" Hamiltonian corresponding to $V = 0$; only the constant electric field is acting. To state the hypotheses on V we will use the notations $\langle x\rangle = (1 + |x|^2)^{1/2}$ and $x^- = \max\{-x, 0\}$.

Hypotheses. There exist constants $C, \mu_V, \rho_V, R_V, \epsilon_V > 0$, an operator V_0 and function $V_{\mathcal{A}}$ so that $V = V_{\mathcal{A}} + e^{-\mu_V x_1^-} V_0$ and:

1. $V_{\mathcal{A}}(x)$ is in $C^\infty(\mathbb{R}^n)$, real valued and has an analytic extension to the cone $\Gamma_V = \{x \in \mathbb{C}^n : \Re x_1 < -R_V, |\Im x| < \rho_V |\Re x_1|\}$, denoted $V_{\mathcal{A}}$, and $\lim_{|x|\to\infty} V_{\mathcal{A}}(x) = 0$ and

$$|V_{\mathcal{A}}(x)| \le C\langle \Re x_1\rangle^{-\epsilon_V} \text{ for } x \in \Gamma_V \cup \mathbb{R}^n \tag{1}$$

2. V_0 is symmetric and H_0-compact and commutes with all operators which are multiplication by a function of x_1.

Here operators which are functions of x act multiplicatively; $V_{\mathcal{A}}$ and $e^{-\mu_V x_1^-}$ are examples. It is well known that H_0 is a self adjoint operator and V is H_0-compact and so H is also self adjoint on the domain of H_0 (see [6]).

Example. If $V_{\mathcal{A}}(x) = \langle x_1\rangle^{-\epsilon_V}\langle \ln\langle x\rangle\rangle^{-1}$ and if $V_0 \in L^p(\mathbb{R}^n)$ for $p > \max\{2, n/2\}$ or if $V_0 \in L^\infty(\mathbb{R}^n)$ and $\lim_{|x|\to\infty} V_0(x) = 0$ then the Hypothesis is satisfied.

The wave operators of scattering theory that are most convenient at present are the *two Hilbert space wave operators*

$$W_J^\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} J^\pm e^{-itH_0}\,,$$

where "s-lim" indicates that the limit is taken in the strong operator topology and where $J^\pm$ are some bounded operators on $L^2(\mathbb{R}^n)$. If V is "short range" (roughly $\epsilon_V > 1/2$ if $F > 0$ and $V(x) = O(\langle x\rangle^{-1-\epsilon})$ for some $\epsilon > 0$ if $F = 0$) then the wave operators are known to exist and be complete [1, 3] when $J^\pm$ is the identity operator $J^\pm = \mathbf{1}$ (these wave operators $W_{\mathbf{1}}^\pm$ are known as the *Møller wave operators*). However if V is not short range then it is not possible to choose $J^\pm$ to be the identity [4, 8]. In both the long range and short range case the choice of $J^\pm$ is: (these are the "time independent modifiers" of Isozaki, Kitada and Yajima)

$$J^\pm u(x) = (2\pi)^{-n/2} \int e^{i\phi^\pm(x,\xi)} a^\pm(x,\xi)\hat{u}(\xi)\, d\xi\,, \tag{2}$$

where $\hat{u}$ denotes the Fourier transform of u and $u \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of smooth functions of rapid decrease and where integrals are over $\mathbb{R}^n$ unless otherwise indicated. The phase $\phi^{\pm}$ and symbol $a^{\pm}$ are chosen so that the symbol of $HJ^{+}-J^{+}H_0$ (resp. $HJ^{-}-J^{-}H_0$) falls off exponentially fast in the outgoing (resp. incoming) region of space. Details about the construction of $J^{\pm}$, that is the choice of $a^{\pm}$ and $\phi^{\pm,}$, will be given elsewhere [10]. Historically the wave operators first used to study long range potential scattering were the *modified wave operators* introduced by J.D. Dollard [4].

$$\Omega_{\gamma}^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}e^{-i\gamma_t(D)}\,, \tag{3}$$

where $\gamma_t(\xi)$ is a real valued function on $\mathbb{R}^n \ni \xi$ to be chosen and $D = -i\nabla$ so that $e^{-i\gamma_t(D)}$ is Fourier equivalent to a multiplication operator.

The *scattering operator* is defined in terms of the wave operators

$$S_J = (W_J^{+})^{*}W_J^{-} \quad \text{and} \quad \Sigma_{\gamma} = (\Omega_{\gamma}^{+})^{*}\Omega_{\gamma}^{-}\,.$$

It is known that the wave operators $W_J^{\pm}$ exist and are "complete" for the appropriate choice of $J^{\pm}$ and similarly for $\Omega_{\gamma}^{\pm}$ for some choice of γ (White [7, 8]). Significantly, for the present purposes, it follows that S_J and Σ_{γ} exist and are unitary on $L^2(\mathbb{R}^N)$. It is known in general that the scattering operator (S_J or Σ_{γ}), commutes with H_0 by the Intertwining Principle Consequently it is possible to restrict S_J and Σ_{γ} to manifolds of "free" energy $H_0 = \lambda$: the restrictions are the *scattering matrix* and are denoted $S_J(\lambda)$ and $\Sigma_{\gamma}(\lambda)$ respectively. The scattering matrix maps $\mathbb{R} \ni \lambda$ to the space of unitary operators on (a space isomorphic to) $L^2(\mathbb{R}^{n-1})$. For comparison, in the Schrödinger case, the scattering matrix maps $[0,\infty)$ with values in the space of unitary operators on $L^2(\mathbf{S}^{n-1})$, where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.

Some definitions and notation is needed to state the results. Choose $\chi^{+} \in C^{\infty}(\mathbb{R})$ and $\kappa > 0$ so that $\chi^{+}(\xi_1) = 1$ if $x_1 > \kappa$ and $\chi^{+}(\xi_1) = 0$ if $x_1 < -\kappa$ and so that $\chi^{+}(\xi_1)^2 + \chi^{+}(-\xi_1)^2 = 1$ and $\chi^{+}(\xi_1) \geq 0$. Define $\chi^{-}(\xi_1) = \chi^{+}(-\xi_1)$. Further define an operator on $L^2(\mathbb{R}^n)$

$$[\chi^{\pm}(D_1)u]\hat{}(\xi) = \chi^{\pm}(\xi_1)\hat{u}(\xi)\,,$$

where $\hat{u}$ denotes Fourier transform. Then $\chi^{+}(D_1)u$ (resp. $\chi^{-}(D_1)u$) is an outgoing (resp. incoming) portion of u. $\chi^{+}(D_1)$ is not a projection operator but if we replaced χ^{+} by the Heaviside function it would be and so intuitively $\chi^{+}(D_1)$ approximates a projection onto outgoing states. The parameter $\kappa > 0$ is fixed adequately large; it plays a technical role in the proof of Theorem 1 below, to assure that certain commutators are small. We define further the effective potential $V_J = V_J^{+} + V_J^{-}$ where $V_J^{\pm} = HJ^{\pm}\chi^{+}(D_1) - J^{\pm}\chi^{+}(D_1)H_0$. The first result concerns the resolvent operator $R(z) = (H - z)^{-1}$.

Theorem 1. *For any $\mu > 0$, each of the 9 operators*

$$e^{-\mu x_1^-} R(z) e^{-\mu x_1^-}, \quad e^{-\mu x_1^-} R(z) V_J^{\pm}, \quad (V_J^{\pm})^* R(z) e^{-\mu x_1^-},$$
$$(V_J^{\pm})^* R(z) V_J^{\pm} \text{ and } (V_J^{\mp})^* R(z) V_J^{\pm}$$

has a meromorphic extension from $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) to $\mathbb{C}$ as a bounded operator on $L^2(\mathbb{R}^n)$.

The extension of $R(z)$ established by this theorem is of interest because mathematicians define a *resonance* of the quantum system to be a pole of the extension. Physicists, however, define a resonance as a pole of the meromorphic continuation of the scattering matrix. A drawback of this definition from the mathematician's viewpoint is that the scattering matrix was not known to have such a continuation in general. Our next result establishes the existence of a continuation for all V satisfying the Hypothesis. To state it we first introduce a standard spectral representation of the H_0.

Define therefore $G(\xi) = \xi_1(\xi_2^2 + \ldots + \xi_n^2) + \xi_1^3/3$ and also the operator U on $L^2(\mathbb{R}^n) \ni v$

$$Uv(x) = F^{-n/2} \left[e^{iG(D)/F} v \right] (x/F). \tag{4}$$

Then U is a unitary operator on $L^2(\mathbb{R}^n)$ and $H_0 = U^* x_1 U$. Now suppose $v \in \mathcal{S}(\mathbb{R}^n)$ and define, for each real λ, $T_0(\lambda)v(x_\perp) = Uv(\lambda, x_\perp)$, that is U, followed by restriction to the plane $x_1 = \lambda$ so that $T_0(\lambda)v$ is in $\mathcal{S}(\mathbb{R}^{n-1})$. We may now state the main result of this paper.

Theorem 2. *For almost every real λ*

$$S_J(\lambda) - 1 = +2\pi i [T_0(\lambda) V_J^* R(\lambda + i0) V_J T_0(\lambda)^* - T_0(\lambda) J^* V_J T_0(\lambda)^*] \tag{5}$$

and $S_J(\lambda)$ has a meromorphic extension in λ to all of $\mathbb{C}$ as a bounded operator on $L^2(\mathbb{R}^{n-1})$. Moreover $T_0(\lambda) J^* V_J T_0(\lambda)^*$ has a holomorphic extension to $\mathbb{C}$.

Consequently, a pole of the (extended) scattering matrix $S_J(\lambda)$ must be a pole of the (extended) resolvent $R(z)$. That is to say a resonance from the physicists' perspective is also a resonance from the mathematicians' perspective. The converse is not clear. In the case of Schödinger operators ($F = 0$), the two concepts of resonance coincide, see [5]. This theorem is somewhat similar to the comparable Schödinger result [2], [5] but the differences are remarkable. Under hypotheses comparable to those here the scattering matrix has an extension to a cone in the complex plane but, in the long range case the extension exists as an operator on certain Gevrey spaces and not simply on L^2 as here.

We consider next the modified wave operators Ω_γ. the definition of γ can be stated at least in terms of the phase function $\phi^{\pm}$ in the definition (2) of $J^{\pm}$.

Let $\theta^{\pm}(x,\xi) = \phi^{\pm}(x,\xi) - x \cdot \xi$, then we define

$$\gamma_t(\xi) = \mp\theta^{+}(-R_\gamma \mathbf{e}_1 \pm 2t\xi - Ft^2\mathbf{e}_1, \pm\xi - Ft\mathbf{e}_1) \tag{6}$$

if $\pm t > 0$ and where $R_\gamma > 0$ is some constant. Our final result can now be stated (for proofs of all results here, see White [10]).

Theorem 3. *The modified wave operators $\Omega_\gamma^{\pm}$ of (3) where the modifier is defined by (6) exist and $\Omega_\gamma^{\pm} = W_J^{\pm}$. Consequently the two definitions of scattering matrix S_J and Σ_γ coincide and they are unitary and so the two definitions of the scattering matrix also coincide: $\Sigma_\gamma(\lambda) = S_J(\lambda)$. The scattering matrix has a meromorphic extension to the entire complex plane as a bounded operator on $L^2(\mathbb{R}^{n-1})$. If $\epsilon_V > 1/2$ then the Møller wave operators $W_{\mathbf{1}}^{\pm}$ exist and $W_{\mathbf{1}}^{\pm} = W_J^{\pm} = \Omega_\gamma^{\pm}$.*

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