International Journal of Pure and Applied Mathematics
Volume 42 No. 2 2008, 303-308

LEFT INVARIANT OPTIMAL CONTROL SYSTEMS AND SUB-RIEMANNIAN GEOMETRY

Felipe Monroy-Pérez[1 §], Alfonso Anzaldo-Meneses[2]

[1]Mathematical Analysis Research Area
Universidad Autonoma Metropolitana – Azcapotzalco
Av. San Pablo 180, Azcapotzalco, México D.F., 022000, MÉXICO
e-mail: fmp@correo.azc.uam.mx

[2]Theoretical Physics Research Area
Universidad Autonoma Metropolitana – Azcapotzalco
Av. San Pablo 180, Azcapotzalco, México D.F., 022000, MÉXICO
e-mail: alfons_rex@hotmail.com

Abstract: In this paper we present the optimal control problem consisting in the minimization of the functional $\int u_1^2 + \cdots + u_n^2$ among the solutions $t \mapsto (g, u_1, \ldots, u_n)$ of the control system $\dot{g} = u_1X_1 + \cdots u_nX_n$, $g \in G$ and $u_i \in L^2(\mathbb{R})$. Here G is a Lie group, and $\Delta = \{X_1, \ldots, X_n\}$ with $n < \dim G$, is a family of left invariant vector fields on G. This is a generalization to the Lie-group theoretical framework of the classical affine-in-controls-quadratic optimal control problem. We establish the problem as the sub-Riemannian geodesic problem defined on G by Δ and a smooth varying metric defined on the planes $\Delta(g)$.

AMS Subject Classification: 49K30, 49Q99, 93B29
Key Words: optimal control, Lie groups and algebras, sub-Riemannian geodesics, Hamiltonian system, extremal curves

1. Introduction

Optimal control theory on Lie groups finds its roots in the pioneering papers

Received: August 17, 2007

[§]Correspondence author

by R.W. Brockett [2] and the one by V. Jurdjevic and H. Sussmann [5]. The Lie group setting in optimal control provides a very robust theory for the deep understanding of both theoretical problems and practical applications, a nice recollection of the geometric methods in control theory can be found in the recent book by A. Agrachev and Y. Sachkov [1].

Sub-Riemannian geometry has attracted recently the attention in different branches of mathematics. The first example of a sub-Riemannian structure was provided by R.W. Brockett in [3]. It consist of the structure in $\mathbb{R}^3$ given by the distribution of two vector fields $X_1 = \partial x + y\partial z, \quad X_2 = \partial y - x\partial z$. The vector fields $\{X_1, X_2, X_3 := [X_1, X_2]\}$ are left invariant with respect Heisenberg multiplication $(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1+x_2, y_1+y_2, z_1+z_2+(x_1y_2-x_2y_1))$, and for all $q \in \mathbb{R}^3$ one has that $\text{span}\{X_1, X_2, X_3\}_q = \mathrm{T}_q\mathbb{R}^3$. Furthermore, for any absolutely continuous curve $t \mapsto q(t)$ satisfying $\dot{q}(t) \in \text{span}\{X_1(q(t)), X_2(q(t))\}$, there exist $u, v \in L_2(\mathbb{R})$ such that $\dot{q}(t) = uX_1(q(t)) + vX_2(q(t))$, a.e. which naturally defines a left invariant control system in the Heisenberg group.

The first formal presentation of sub-Riemannian geometry was given by R. Strichartz in [8], after that abundant literature has been written on the topic, a general overview was presented by I. Kupka at the Bourbaki seminar [6], and an authoritative presentation is given by M. Gromov in [4] as well as by R. Montgomery in [7].

Apart from this introduction, this paper contains 5 sections. In Section 2 we discuss the basic definitions on sub-Riemannian geometry on arbitrary Lie groups. In Section 3 we set the sub-Riemannian geodesic problem for step two distributions satisfying certain Lie bracket relations. In Section 4, we apply the Pontryagin Maximum Principle for deriving necessary conditions for geodesics. In Section 5 we carry out an explicit integration process for the extremals, exhibiting a complete set of integral of motion in involution. Finally in Section 6, we discuss, some low dimensional cases.

2. Sub-Riemannian Geometry on Lie Groups

Let G be a Lie group and let Δ be finite family of left invariant vector fields. If one defines $\Delta^2 = [\Delta, \Delta]$ and $\Delta^j = [\Delta, \Delta^{j+1}]$, for $j = 3, \ldots$, then for each $g \in G$, one obtains a flag of linear subspaces $\Delta(g) \subset \Delta^2(g) \subset \Delta^3(g) \subset \cdots \subset T_gG$.

Definition 2.1. The distribution Δ is said to be bracket generating if for each $g \in G$ there is a $\eta \in \mathbb{N}$ such that $\Delta^\eta = T_gG$.

The sub-Riemannian structure on G, is defined by means of a smoothly varying inner product $\langle\cdot,\cdot\rangle_{\Delta(g)}$ on the space $\Delta(g)$. An absolutely continuous curve $g:[0,T_g]\to G$ is by definition $\Delta-$*horizontal* or *admissible*, provided $\dot g(t)\in\Delta(g)$, almost everywhere. The sub-Riemannian length of a $\Delta-$horizontal curve $g(t)$ is then defined as $\ell(g)=\int_0^{T_g}\|\dot g(t)\|\ dt$. The fact that the left invariant distribution Δ is bracket-generating guarantees that any two elements in G can be connected by a $\Delta-$horizontal curve (Chow-Rashevskii's Theorem). The sub-Riemannian geodesic problem consists then in the minimization of the functional ℓ in the class of $\Delta-$horizontal curves.

3. Geodesics for Step Two Distributions

Assume that G is a nilpotent Lie group of dimension $n(n+1)/2$, we assume also that Δ is a left invariant, bracket generating distribution on G, for which a *nilpotent basis* $\{X_1,\dots,X_n\}$ with order of nilpotency one is given, that is, $\Delta(g)=\operatorname{span}\{X_1(g),\dots,X_n(g)\}$ and $\operatorname{ad}^k_{X_i}(X_j)=0$, for all $k>1$. Furthermore, assume that the vector fields $X_1,\dots,X_n$ together with the non-zero Lie brackets $X_{ij}:=[X_i,X_j]$, $1\le i<j=2,\dots,n$, determine a basis of left invariant vector fields for the Lie algebra $\mathfrak{g}$ of the group G. The sub-Riemannian structure on G is defined by declaring the vectors $X_1(g),\dots,X_n(g)$ orthonormal, for each $g\in G$. The geodesic problem is equivalent to the optimal control with dynamics

$$\dot g(t)=u_1(t)X_1(g(t))+\cdots+u_n(t)X_n(g(t)),\tag{1}$$

with measurable and bounded admissible controls $\mathcal{U}:=\{\mathbf{u}=(u_1,\dots,u_n)\}$ and cost functional

$$\Lambda(g,\mathbf{u})=\frac{1}{2}\int(u_1^2+\cdots+u_n^2)\ dt.$$

4. Necessary Conditions for Geodesics

If H_i denotes the Hamiltonian corresponding to X_i, and H_{ij} the one corresponding to X_{ij}, then the dual variable p can be represented by a pair $(\mathbf{h},\mathbf{H})\in\mathbb{R}^n\times so_n$, with $\mathbf{h}=(H_1,H_2,\dots,H_n)$ and $\mathbf{H}=(H_{ij})$. Each $\mathbf{u}\in\mathcal{U}$ determines a Hamiltonian function

$$\mathcal{H}_{\lambda,\mathbf{u}}=-\frac{\lambda}{2}(u_1^2+\cdots+u_n^2)+u_1H_1+\cdots+u_nH_n,$$

where the parameter λ is normalized to take the values 1 (normal) or 0 (abnormal). The following is an invariant version of the classical Pontryagin Maximum Principle.

Theorem 4.1. *A solution curve $t \mapsto (g(t), \widehat{\mathbf{u}})$ of system (1), is Λ-optimal, if it is the projection of an extremal curve (g, p) along which the inequality $\mathcal{H}_{\lambda,\widehat{\mathbf{u}}} \geq \mathcal{H}_{\lambda,\mathbf{w}}$ holds a.e., for all $\mathbf{w} \in \mathcal{U}$. Furthermore, for $\lambda = 0$, the dual variable p cannot be identically zero.*

In conclusion, a geodesic $g : [0, T_g] \to G$, $t \mapsto g(t)$ appears always as the first coordinate of a trajectory $t \mapsto (g(t), \widehat{\mathbf{u}}(t))$ of system (1), and it is necessarily projection of an extremal curve $t \mapsto (g(t), p(t)) = (g(t), \mathbf{h}(t), \mathbf{H}(t))$. We consider here only the normal case, $\lambda = 1$. In this case we have $\mathcal{H}_{1,\mathbf{u}} = -\frac{1}{2}(u_1^2 + \cdots + u_1^2) + u_1 H_1 + \cdots + u_n H_n$, and the optimality condition of the maximum principle yields $u_1 = H_1, \cdots, u_n = H_n,$, along the extremal, therefore the system Hamiltonian is quadratic

$$\mathcal{H} = \frac{1}{2}\left(H_1^2 + \cdots + H_n^2\right),$$

and differentiating the $H_{i's}$ along the extremal one gets

$$\begin{aligned}
\dot{H}_1 &= \{H_1, \mathcal{H}\} = H_2 H_{12} + H_3 H_{13} + \cdots + H_n H_{1n}\,, \\
\dot{H}_2 &= \{H_2, \mathcal{H}\} = H_1 H_{21} + H_3 H_{23} + \cdots + H_n H_{2n}\,, \\
&\vdots \\
\dot{H}_n &= \{H_n, \mathcal{H}\} = H_1 H_{n1} + H_2 H_{n2} + \cdots + H_{n-1} H_{(n-1)n},
\end{aligned}$$

or equivalently $\dot{\mathbf{h}} = \mathbf{H}\ \mathbf{h}$. But then, since the H_{ij} are central elements, the skew-symmetric matrix $\mathbf{H}$ remains constant along the extremal. In fact

$$\begin{aligned}
\dot{H}_{ij} &= \{H_{ij}, \mathcal{H}\} = \frac{1}{2}\{H_{ij}, H_1^2 + \cdots + H_n^2\} \\
&= H_1\{H_{ij}, H_1\} + \cdots + H_n\{H_{ij}, H_n\} = 0.
\end{aligned}$$

In summary, we can write the following result for the normal case:

Theorem 4.2. *If $(g(t), \mathbf{h}(t), \mathbf{H}(t))$ is a normal extremal, then*

$$\frac{dg}{dt} = H_1 X_1(g) + \cdots + H_n X_n(g)\,, \tag{2}$$

$$\frac{dp}{dt} = (\dot{\mathbf{h}}, \dot{\mathbf{H}}) = (\mathbf{Hh}, \mathbf{0})\,. \tag{3}$$

5. Integration of the Extremal Equations

As customary one has that the system Hamiltonian $\mathcal{H}$ is constant along the extremal. In fact, a direct differentiation along the extremal yields $\dot{\mathcal{H}} = H_1\{H_1,\mathcal{H}\} + \cdots + H_n\{H_n,\mathcal{H}\} = \mathbf{h}^T\mathbf{H}\mathbf{h} = 0$, since $\mathbf{H}$ is skew-symmetric. Furthermore, the n initial conditions $\mathbf{h}(0) = (H_1(0),\ldots,H_n(0))$, together with the $n(n-1)/2$ constants $H_{k_{ij}}$ provide a complete set of integrals of motion for the system (3) that guarantee the integrability of the system.

Theorem 5.1. *The adjoint system (3) is integrable by quadratures and its integral curves are given by*

$$\mathbf{h}(t) = e^{t\mathbf{H}}\,\mathbf{h}_0.$$

Since $\mathbf{H}$ is skew-symmetric its eigenvalues are purely imaginary and they appear in $\pm$ pairs or are zero, then matrix $e^{t\mathbf{H}}$ can be explicitly calculated.

6. Low Dimensional Cases

The (4,10)-Case. In this case $\mathbf{H}$ is a 4×4 matrix with the following eigenvalues $\{\lambda_1,-\lambda_1,\lambda_2,-\lambda_2\}$, and

$$\lambda_{1,2}^2 = \frac{1}{4}\mathrm{trace}(\mathbf{H}^2) \pm \sqrt{(\frac{1}{4}\mathrm{trace}(\mathbf{H}^2))^2 - \det(\mathbf{H})},$$

with

$$\mathrm{trace}(\mathbf{H}^2) = 2\lambda_1^2 + 2\lambda_2^2 = -2(H_{12}^2 + H_{13}^2 + H_{14}^2 + H_{23}^2 + H_{24}^2 + H_{34}^2),$$

and

$$\det(\mathbf{H}) = \lambda_1^2\lambda_2^2 = (H_{14}H_{23} - H_{13}H_{24} + H_{12}H_{34})^2.$$

The matrix $\mathbf{H}$ satisfies

$$\mathbf{H}^4 - \frac{1}{2}\mathrm{trace}(\mathbf{H}^2)\,\mathbf{H}^2 + \det(\mathbf{H})I = 0.$$

Now,

$$\mathbf{h}(t)=\frac{1}{\lambda_1^2-\lambda_2^2}\Big(\lambda_1^2\cosh(\lambda_2 t)I - \lambda_2^2\cosh(\lambda_1 t)I + (\frac{\lambda_1^2}{\lambda_2}\sinh(\lambda_2 t) - \frac{\lambda_2^2}{\lambda_1}\sinh(\lambda_1 t))\mathbf{H}$$
$$+(\cosh(\lambda_1 t) - \cosh(\lambda_2 t))\mathbf{H}^2 + (\frac{1}{\lambda_1}\sinh(\lambda_1 t) - \frac{1}{\lambda_2}\sinh(\lambda_2 t))\mathbf{H}^3\Big)\mathbf{h}_0.$$

The (5,15)-Case. This case leads to a singular 5×5 dimensional matrix $\mathbf{H}$, with eigenvalues $\{0,\lambda_1,-\lambda_1,\lambda_2,-\lambda_2\}$, with

$$\lambda_{1,2}^2 = \frac{1}{4}\mathrm{trace}(\mathbf{H}^2) \pm \sqrt{(-3(\frac{1}{4}\mathrm{trace}(\mathbf{H}^2))^2 + \frac{1}{2}\mathrm{trace}(\mathbf{H}^4))}.$$

H satisfies

$$\mathbf{H}(\mathbf{H}^4 + \Lambda_2 \mathbf{H}^2 + \Lambda_4 I) = 0,$$

with the elementary symmetric functions $\Lambda_2 = -\lambda_1^2 - \lambda_2^2 = -\frac{1}{2}\mathrm{trace}(\mathbf{H}^2)$ and $\Lambda_4 = \lambda_1^2\lambda_2^2 = (\frac{1}{2}\mathrm{trace}(\mathbf{H}^2))^2 - \frac{1}{2}\mathrm{trace}(\mathbf{H}^4)$. It follows that

$$\mathbf{h}(t) = \frac{1}{(\lambda_1^2 - \lambda_2^2)}\Bigg(I + \{-\frac{\lambda_2^2}{\lambda_1}\sinh(\lambda_1 t) + \frac{\lambda_1^2}{\lambda_2}\sinh(\lambda_2 t)\}\mathbf{H} +$$

$$\{\frac{\lambda_2^2}{\lambda_1^2}(1 - \cosh(\lambda_1 t)) - \frac{\lambda_1^2}{\lambda_2^2}(1 - \cosh(\lambda_2 t))\}\mathbf{H}^2 + \{\frac{1}{\lambda_1}\sinh(\lambda_1 t) - \frac{1}{\lambda_2}\sinh(\lambda_2 t)\}\mathbf{H}^3$$

$$+\{\frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2} + \frac{1}{\lambda_1^2}\cosh(\lambda_1 t) - \frac{1}{\lambda_2^2}\cosh(\lambda_2 t)\}\mathbf{H}^4\Bigg)\mathbf{h}_0.$$

References

[1] A. Agrachev, Y. Sachkov, *Control Theory from the Geometric Viewpoint*, Encyclopedia dia of Mathematical Sciences, Springer-Verlag (2004).

[2] R.W. Brockett, System theory on group manifolds and coset spaces, *SIAM J. Control*, **10** (1972), 265-284.

[3] R.W. Brockett, Control theory and singular Riemannian geometry, *New Directions in Applied Mathematics* (Ed-s: P.J. Hilton, G.S. Young), Springer-Verlag (1981).

[4] M. Gromov, Carnot-Caratheodory spaces seen from within, In: *Sub-Riemannian geometry* (Ed-s: A. Bellaïche, J.-J. Risler), Birkhäuser (1996).

[5] V. Jurdjevic, H.J. Sussmann, Control systems on Lie groups, *J. Diff. Equations*, **12** (1972), 313-329.

[6] I. Kupka, Géometrie sous-Riemanniene, *Séminaire Bourbaki*, 48éme année (1995-96), No. 817, 1-30.

[7] R. Montgomery, *A Tour of Subriemannian Geometries, their Geodesics and Applications*, Mathematical Surveys and Monographs, Vol. 91, American Mathematical Society (2002).

[8] R.S. Strichartz, Sub-Riemannian geometry, *J. Diff. Geom.*, **24** (1986), 221-263.