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HIGH ORDER COMPACT AND UPWIND METHODS

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Abstract: We consider two types of high order numerical methods for solving the Navier-Stokes equations: compact and upwind methods. Firstly, we analyze procedures for obtaining compact fourth order method to the steady 2D Navier-Stokes equations in the context of stream function-vorticity formulations. Results are compared with those obtained using second order central differences to moderate Reynolds numbers. Secondly, we show a new high order upwind method (called adaptative QUICKEST) for simulating free surface flows at high Reynolds numbers. A crucial point is the discretization of the advective terms in the transport equations, which is of primordial importance for the prediction of the flow problem. Several numerical tests are presented.

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1. Introduction

High order finite difference methods have been used for a long time since they

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have some advantages when compared with second order methods or first order upwind methods. For example, for a given tolerance the number of points employed can be reduced and consequently the CPU time is also reduced. These methods are classified as wide or compact. The wide high order methods are obtained discretizating the equations by fourth order central differences which results a large computational molecule, and the compact methods are constructed with the aims of having narrower molecules and maintaining the high order accuracy. These methods can be obtained for both stream function and stream function-vorticity formulations of the Navier-Stokes equations. To deal with the difficulties to construct compact fourth-order methods to the steady 2D Navier-Stokes equations on a uniform and non-uniform grid Mancera and Hunt [5, 6] have proposed a simple and efficient procedure to obtain compact methods using a computer algebra system, where all substitutions are never done mathematically but are performed at the program which results a short code.

Let be the Navier-Stokes equations in a stream function (ψ) and vorticity (ζ) formulation

$$\nabla^2\psi = -\zeta, \qquad \nabla^2\zeta = Re\left(\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y}\right), \tag{1}$$

where x and y are Cartesian coordinates and Re is the Reynolds number. In order to make these equations more flexible, to deal with highly variable flows, we convert the square cells into rectangles by the transformations $x = f(\xi)$, $y = g(\eta)$. So equations (1) are written, on the new coordinates, as

$$\frac{1}{f'^2}\frac{\partial^2\psi}{\partial\xi^2} - \frac{f''}{f'^3}\frac{\partial\psi}{\partial\xi} + \frac{1}{g'^2}\frac{\partial^2\psi}{\partial\eta^2} - \frac{g''}{g'^3}\frac{\partial\psi}{\partial\eta} = -\zeta, \tag{2}$$

$$\frac{1}{f'^2}\frac{\partial^2\zeta}{\partial\xi^2} - \frac{f''}{f'^3}\frac{\partial\zeta}{\partial\xi} + \frac{1}{g'^2}\frac{\partial^2\zeta}{\partial\eta^2} - \frac{g''}{g'^3}\frac{\partial\zeta}{\partial\eta} = \frac{Re}{f'g'}\left(\frac{\partial\zeta}{\partial\xi}\frac{\partial\psi}{\partial\eta} - \frac{\partial\zeta}{\partial\eta}\frac{\partial\psi}{\partial\xi}\right), \tag{3}$$

with f', f'', g', g'' the derivatives of the functions f and g, respectively. On the formulation of the numerical method, we consider a uniform grid with grid spacing δ_x and δ_y. The stream function and vorticity equations (1) are written as $eqn1 := u_{xx} + u_{yy} + v$ and $eqn2 := v_{xx} + v_{yy} - Re * (u_y * v_x - u_x * v_y)$, where (u, v) is the velocity field. Approximating each derivative in these expressions by second order finite differences, we obtain the approximated equations $eqn1 = a00 - \delta_x{**}2{*}a20 - \delta_y{**}2{*}a02$ and $eqn2 = b00 - \delta_x{**}2{*}b20 - \delta_y{**}2{*}b02$, which give the fourth order numerical method, with a's and b's the method coefficients. This construction is made via the software *Maple* and the substitutions are made inside the main code.

To test the numerical method we have solved an occlusion stepped channel problem with walls at $y = \pm 1$ for $x < m$ and $x > n$, $y = \pm(1/p)$ for $m < x < n$ and $(1/p) \leq |y| \leq 1$ for $x = m$ and $x = n$, where m, n and p are integers, $m < n$. Due to the symmetry, the problem is solved for $y \geq 0$ only. The boundary conditions are given bysa

$$
\begin{aligned}
\psi &= 1, \quad \frac{\partial \psi}{\partial y} = 0 \text{ on } y = 1,\ x \leq m, x \geq n, \text{ and } y = \frac{1}{p}, m < x < n, \\
\psi &= 1, \quad \frac{\partial \psi}{\partial x} = 0 \text{ on } x = 0 \text{ and } x = n, \frac{1}{p} \leq y \leq 1, \\
\psi &= 0, \quad \frac{\partial^2 \psi}{\partial y^2} = 0 \text{ on } y = 0, \\
\psi &\to \frac{3}{2}y - \frac{1}{2}y^3, \quad \zeta \to 3y \quad \text{as} \quad x \to -\infty, \\
\psi &\to \frac{3}{2}y - \frac{1}{2}y^3, \quad \zeta \to 3y \quad \text{as} \quad x \to +\infty,
\end{aligned}
\tag{4}
$$

with Poiseuille flow far upstream and far downstream.

For a fourth order method, the error is calculated by $E_F \simeq (\phi_F - \phi_M)/15$, where ϕ_F is the solution on a grid with half of the number of points of the solution ϕ_M in each direction of the computational axis.

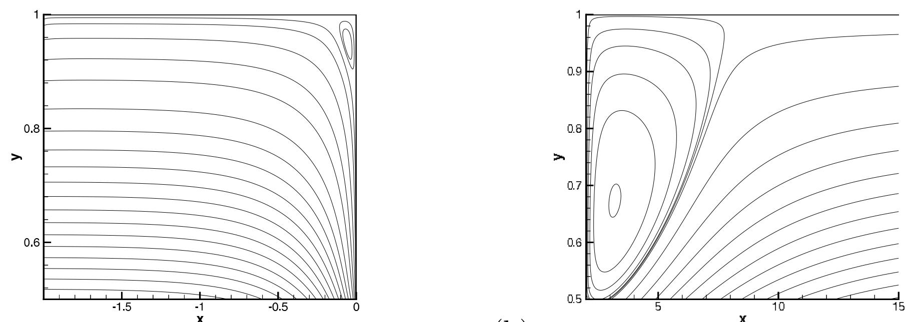
Results are considered on two different grids, to say, 64×32 and 128×64, where the grid size is $\delta_x = 1/nx$ and $\delta_y = 1/ny$, with $nx = 64$ or $nx = 124$ and $ny = 32$ or $ny = 64$, respectively, and are shown in terms of the root-mean-square (rms). As can be seen in Table 1, the errors vary from $\sim 10^{-6}$ to $\sim 10^{-4}$. These errors are smaller than the errors of the second order method, while the ratio between the errors of the second order method and the compact fourth order method is not large, as expected, due to the two singular points. The ratio between the errors decreases as Re increases, but the second order method fails to converge for $Re = 500$. Figure 1 shows the streamlines before and after the occlusion to a non-uniform grid, where $x = f(\xi) = \frac{\Delta x_0}{k} \sinh(k\,\xi)$, and $y = g(\eta) = \eta + \frac{1}{2\pi}(1 - \Delta y_0) \sin(2\pi\eta)$, with $\Delta x_0 = 0.01$, $\Delta y_0 = 0.025$, $k = 2.90049$, and the upstream position at $x \simeq -2$ and the downstream position at $x \simeq 663$. We can observe a fully Poiseuille flow.

2. Upwind Methods

Several upwind schemes are available in the literature for approximating the advective terms. The first order upwind, for instance, is unconditionally bounded, but it can lead to large errors causing the solution grossly inaccurate. On the other hand, the classical Lax-Wendroff, Central and QUICK schemes can provoke spurious oscillations near discontinuities, causing numerical instabilities.

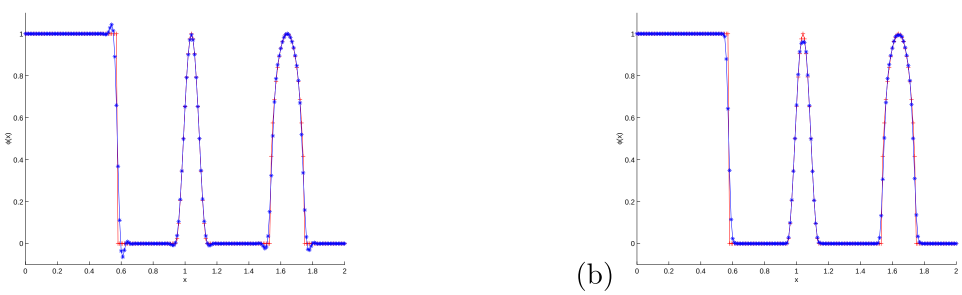
	Methods		
	4th order	2nd order	
Re	ψ errors	ψ errors	Ratio
0	7.61(-6)	2.09(-4)	27.46
1	8.60(-6)	2.10(-4)	24.42
10	2.43(-5)	2.54(-4)	10.45
50	4.29(-5)	3.43(-4)	8.00
100	4.32(-5)	4.18(-4)	9.68
250	2.65(-4)	8.86(-4)	3.34
500	4.35(-4)	*	*

Table 1: Rms errors to the stream function



(a) (b)

Figure 1: Streamlines (a) before and (b) after occlusion with $Re = 100$



(a) (b)

Figure 2: Comparison between numerical and analytical solutions: (a) QUICKEST scheme, and (b) Adaptative QUICKEST scheme

In recent years, efforts have been made to build schemes which can give high

(a) (b)

Figure 3: Numerical simulations of free surface flows at high Reynolds number: (a) Hydraulic jump at Reynolds 100; (b) Fluid structure interaction at Reynolds 99000

order accuracy without introducing spurious oscillations and converge to physically correct solution. Examples include SMART, QUICKEST and CUBISTA schemes. In view of the fact that most fluid flow codes need a more elaborated upwind scheme, in this work it is presented the development/application of a finite difference upwind technique (called adaptative QUICKEST) for convection dominated problems. This scheme is a new bounded high order upwind scheme proposed by Ferreira et al [1] for numerical solution of convection dominated problems.

The adaptative QUICKEST is derived in the context of the normalized variable of Leonard and Niknafs [4] and by enforcing the TVD property of Harten [3]. Consequently, it satisfies CBC criterion of Gaskell and Lau [2]. The main idea in the derivation this scheme is to combine accuracy and monotonicity, while ensuring flexibility. It is based on unsteady analysis of the 1D advection equation and retains the Courant number as a free parameter. It can also ensure that total variation of the variables does not increase with time; thus no spurious numerical oscillations (maxima or minima) are generated. The numerical solution can be second or third order accurate in the smooth parts of the solution, but only first order near regions with large gradients. In summary, this advection scheme is implemented by the functional relationship

$$\hat{\phi}_f=\begin{cases} (2-\nu)\hat{\phi}_U, & 0<\hat{\phi}_U<a\,, \\ \hat{\phi}_U+\frac{1}{2}(1-|\nu|)(1-\hat{\phi}_U)-\frac{1}{6}(1-\nu^2)(1-2\hat{\phi}_U), & a\le\hat{\phi}_U\le b\,, \\ 1-\nu+\nu\hat{\phi}_U, & b<\hat{\phi}_U<1\,, \\ \hat{\phi}_U, & \text{elsewhere}\,, \end{cases} \tag{5}$$

where $\hat{\phi}_{()} = (\phi_{()} - \phi_R)/(\phi_D - \phi_R)$ is the Leonard's normalized variable. The constants a and b in equation (5) are given by,

$$a=\frac{2-3|\nu|+\nu^2}{7-6\nu-3|\nu|+2\nu^2} \quad \text{and} \quad b=\frac{-4+6\nu-3|\nu|+\nu^2}{-5+6\nu-3|\nu|+2\nu^2}.$$

As an example of validation, we consider the 1D wave equation, with $0 \leq x \leq 2$, $0 \leq \phi(x,5) \leq 1.1$, and initial data

$$\phi(x,0) = \begin{cases} 1, & 0 \leq x \leq 0.13\,, \\ \sin^2(5\pi(x+0.11)), & 0.5 \leq x \leq 0.7\,, \\ \sqrt{1 - \frac{(x-1.2)^2}{1.12^2}}, & 1.1 \leq x \leq 1.31\,, \\ 0, & \text{elsewhere.} \end{cases}$$

Figure 2 presents the comparison between the analytical solution and *QUICKEST* and adaptative *QUICKEST* solutions. From this figure, one can see that the adaptative QUICKEST scheme produces the best result.

As an illustrative example, we present in Figure 3 the numerical simulation of free surface flows with moving free surfaces at high Reynolds numbers.

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