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HOLOMORPHIC CHERN-SIMONS AND BF THEORIES ON SUPER TWISTOR SPACES

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Abstract: In 2003, Witten[12] introduced the twistor string which is a string theory in super Twistor space, $\mathbf{CP}^{3|4}$. One of the key ideas behind the twistor string is holomorphic Chern-Simons theory. It is of interest to extend the idea of twistor strings and thus holomorphic Chern-Simons theory beyond $\mathbf{CP}^{3|4}$ to other spaces in twistor theory such as super ambitwistor spaces.

In this talk, we shall begin with an introduction to twistor and ambitwistor spaces. We quickly review various results from twistor theory such as Penrose-Ward transforms. We also present various ideas from supergeometry which we will be needing. After a short introduction to Chern-Simons theory and its holomorphic analog, we discuss holomorpic Chern-Simons theory on $\mathbf{CP}^{3|4}$. BF theory, another topological gauge theory and its extension by A. Popov [7] to holomorphic BF theory are reviewed. We also give its extension to complex supermanifolds. Finally, we investigate holomorphic BF theory on super ambitwistor spaces and the role of holomorphic and almost complex bundles in holomorphic Chern-Simons and BF theories.

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1. Minkowski Space and Spinor Notation

We start with Minkowski space, $\mathbf{M^4}$, i.e. $\mathbf{R^4}$, with the metric,

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$$dt^2 - dx^2 - dy^2 - dz^2 \quad .$$

The coordinates are analytically continued so that we are in $\mathbf{C^4}$. We also make a change of coordinates so that

$$\begin{pmatrix} x^{0\dot{0}} & x^{0\dot{1}} \\ x^{1\dot{0}} & x^{1\dot{1}} \end{pmatrix} = \begin{pmatrix} t-z & x-iy \\ x+iy & t+z \end{pmatrix} .$$

Heuristically, the metric in the new coordinates is

$$det \begin{pmatrix} dx^{0\dot{0}} & dx^{0\dot{1}} \\ dx^{1\dot{0}} & dx^{1\dot{1}} \end{pmatrix} = dx^{0\dot{0}} \odot dx^{1\dot{1}} - dx^{1\dot{0}} \odot dx^{1\dot{1}} .$$

This leads us to consider the tangent bundle of complexified Minkowski space as a tensor product of a pair of 2 (complex) dimensional bundles, the spinor bundles: $T\mathbf{M_C} = \mathbf{S}_+ \otimes \mathbf{S}_-$. From above, we also see that a tangent vector is null if and only if it is a simple tensor product, $v^A \otimes u^{\dot{A}}$. The conformal compactification of $\mathbf{M_C}$ is $Gr(2, \mathbf{C^4})$, the Grassmanian of two dimensional subspaces of $\mathbf{C^4}$. A standard coordinate chart in $Gr(2, \mathbf{C^4})$ should now be familiar: $\left(\begin{smallmatrix} 1 & 0 & x^{0\dot{0}} & x^{0\dot{1}} \\ 0 & 1 & x^{1\dot{0}} & x^{1\dot{1}} \end{smallmatrix}\right)$.

2. Twistor Space

Consider the plane in $\mathbf{M}$ generated by varying $\lambda^{\dot{A}}$ in: $x^{A\dot{A}} + v^A\lambda^{\dot{A}}$. It is isotropic: any tangent vector to the plane is null and any two tangent vectors have zero scalar product. Such a plane generated above is called an *α-plane*. Twistor space is the space of α-planes. In the above expression, fixed $x^{A\dot{A}}$ and v^A give an α-plane. However, we can change $x^{A\dot{A}}$ but remain on the same α-plane. If we contract with v_A (where we lowered indices using the antisymmetric, ϵ_{AB}), we get a better parameter for the α-plane: $u^{\dot{A}} = v_A x^{A\dot{A}}$ (using Einstein's summation notation). We thus get coordinates on twistor space: $u^{\dot{A}}, v_A$. Scaling $u^{\dot{A}}$ and v_A by the same amount, gives the same α-plane. Thus, these are homogeneous coordinates: $[u^{\dot{A}}, v_A]$. For $\mathbf{M^4}$, twistor space is $\mathbf{CP^3}$.

This is a case of the Klein correspondence. Our construction gives the correspondence of $Gr(2, \mathbf{C^4})$ with $\mathbf{CP^3}$ (see [5]). This is illustrated in the double fibration:

$$\begin{array}{ccc} & F(1,2,\mathbf{C^4}) & \\ \swarrow & & \searrow \\ \mathbf{CP^3} & & Gr(2,\mathbf{C^4}) \end{array}$$

3. The Ward Correspondence

Recall that a connection on a vector bundle, E, is an extension of the exterior derivative, d , on functions on the base space to a differentiation, D, of sections of the bundle. Locally, $D = d + A$, where A is a local section of the Lie algebra bundle of E tensored with the bundle of 1-forms. The curvature of D is $F_A = D^2 = dA + A \wedge A$.

To write the Yang-Mills equations, we need the Hodge star operator, $*$. For a four dimensional manifold with metric, the Hodge star operator, $* : \bigwedge^2 M \to \bigwedge^2 M$ is defined so that for a 2-form, ω, we have $\omega \wedge *\omega$ is the volume form of M. This can be extended to bundle valued 2-forms. The Yang-Mills equations are then $D_A F_A = 0$ and $D_A * F_A = 0$. The first equation, the Bianchi identity, is satisfied by all connections. For a four dimensional manifold, $*^2 = 1$. There are thus two eigenvalues, 1 and -1. The self-dual and anti self-dual Yang-Mills equations are respectively: $*F_A = F_A$ and $*F_A = -F_A$. Solutions to either of these also satisfy the full Yang-Mills equations.

The α-planes in the construction of twistor space are in fact self-dual. These are planes in which the 2-forms on the plane are entirely self-dual. Similiarly the β-planes are anti self-dual. Thus, the curvature of an anti self-dual connection will vanish when restricted to an α-plane. Over each α-plane we obtain a vector space of covariantly constant sections. There is then a vector space over each point in twistor space, i.e. a vector bundle. Heuristically, this was a natural construction so the bundle will be holomorphic. This bundle will be trivial when restricted to certain "normally" embeded $\mathbf{CP_1}$. These $\mathbf{CP_1}$'s are the left hand projection into $\mathbf{CP_3}$ of the right hand fibres in the above double fibration.

There is a reverse construction of an anti self-dual connection over $\mathbf{M}$ given a holomorphic vector bundle over $\mathbf{CP_3}$ (see [9]). A solution to the anti self-dual Yang-Mills equation (up to gauge equivalence) on $\mathbf{M}$ then corresponds in a one-to-one fashion to a holomorphic vector bundle (up to equivalence of holomorphic bundles) over $\mathbf{CP_3}$.

4. Ambitwistor Space

The ambitwistor space of a complex conformal manifold is its space of null geodesics, see [3]. For $\mathbf{M} = \mathbf{Gr(2, C^4)}$, ambitwistor space is the five dimensional flag manifold, $F(1, 3, \mathbf{C^4})$. This can be embedded in $\mathbf{CP_3} \times \mathbf{CP_3}$. If u_α and v^β are homogeneous coordinates, then $F(1, 3, \mathbf{C^4})$ is the locus of $u_\mu v^\mu = 0$.

Isenberg, Yaskin, and Green [2] showed a Ward correspondence between general solutions to the Yang-Mills equations on $\mathbf{M}$ and vector bundles over the third infinitesmal neighborhood (or *thickening*) of ambitwistor space, sitting inside $\mathbf{CP_3} \times \mathbf{CP_3}$. Witten [10] also produced a related Ward correspondence. His ideas involved supergeometry.

5. Supergeometry

A supermanifold is defined as a ringed space, (X, A), where X is a topological manifold and A is a sheaf of $\mathbf{Z_2}$-graded rings. Let Nil denote the nilpotent ideal of A. We also require that $(X, A/Nil)$ is a differentiable (complex) manifold and $Nil/(Nil)^2$ is a differentiable (holomorphic) vector bundle over $(X, A/Nil)$ (see [4]). Informally, this means there are local even coordinates, x^a, and odd coordinates, ϕ^i . Even coordinates commute with all other coordinates and odd coordinates anti-commute with odd coordinates. Note, in particular, $(\phi^i)^2 = 0$. The dimension of the supermanifold is denoted $n|m$, where n is the number of even and m is the number of odd coordinates. For an introduction to supergeometry, see [5].

Integration on a superspace of one fermionic coordinate is defined by $\int d\phi\, \phi = 1$ and $\int d\phi = 0$. In this notation, $d\phi$ is an integral form and not a differential 1-form. The volume form is a section of the *Berezinian* bundle. Integration in higher dimensions is carried out in the usual manner supplemented with the equations above for integration over odd variables.

In 1977 Witten [10] proposed that solutions to the $N = 3$ supersymmetric Yang-Mills equations correspond to certain holomorphic vector bundles over the *space of super light rays*. This space of super light rays is a super version of ambitwistor space.

6. Holomorphic Chern-Simons and BF theories on a Supermanifold

The Chern-Simons functional on a three dimensional manifold, X, is

$$\int_X \mathrm{tr}\,(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \quad ,$$

where A is a connection on a vector bundle over X. The Euler-Lagrange equations are: $dA + A \wedge A = 0$.

For holomorphic Chern-Simons theory, the action is defined on a three complex dimensional manifold, X, which is Calabi-Yau. There is then a global holomorphic $3,0$ form, Ω. The functional is

$$\int_X \Omega \wedge \mathrm{tr}\,(A^{0,1} \wedge \overline{\partial} A^{0,1} + \frac{2}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1})\,.$$

Here we are taking the 0,1 part of the connection A. The Euler-Lagrange equations are:

$$\overline{\partial} A^{0,1} + A^{0,1} \wedge A^{0,1} = 0\,.$$

The 0, 1 part of the connection defines an almost complex structure on the total space of the bundle. The equation of motion above is just the integrability condition for this almost complex structure to give a holomorphic vector bundle.

Witten [11] observed that holomorphic Chern-Simons perturbation theory is equivalent to a particular open string theory. Witten [10] also observed that $\mathbf{CP}^{3|4}$ is super Calabi-Yau although $\mathbf{CP}^3$ is not. The action of holomorphic Chern-Simons theory can thus be used on super twistor space. Combining this with the Penrose-Ward transform, Witten's twistor string program realizes self-dual Super Yang-Mills theory on four dimensional Minkowski space as a string theory in a super twistor space $\mathbf{CP}^{3|4}$.

Recall, that Penrose-Ward transforms for the *full* Yang-Mills equations involved bundles over either thickened or super *ambitwistor* space. Is there twistor string theory for ambitwistor spaces? Mason and Skinner [6] have created a twistor string theory on an ambitwistor space which is a super CR manifold. Popov and Samann [8] have also proposed a holomorphic Chern-Simons theory for L the super ambitwistor space for $D = 4$, $N = 3$ super Minkowski space. They propose the action:

$$\int_L \Omega \wedge \mathrm{tr}\,(A^{0,1} \wedge \overline{\partial} A^{0,1} + \frac{2}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1}) \wedge \omega^{0,2} \quad ,$$

where $\omega^{0,2}$ is nowhere vanishing and partially closed: $A^{0,1} \wedge \bar{\partial}\omega^{0,2} = 0$.

Is there twistor string theory for when L, is the $17|8$ dimensional super ambitwistor space for a ten dimensional space? One possibility is to use integral forms which are sections of $E_{p,q} = Ber(L) \otimes \bigwedge^{n-p,n-q} TL$. We also have the operator $\bar{\partial}' : E_{p,q} \to E_{p,q+1}$ which is defined as the adjoint operator of $\bar{\partial}$ on the de Rham complex of super forms (see [1]). We can then write an action for the $17|8$ dimensional space as:

$$\int_L \omega \wedge \mathrm{tr}\,(A^{0,1} \wedge \overline{\partial} A^{0,1} + \frac{2}{3} A^{0,1} \wedge A^{0,1} \wedge A^{0,1}) \quad ,$$

where ω is a $\bar{\partial}'$ closed nowhere vanishing section of $E_{17,14}$. Whether such integral forms exist is uncertain (see [8]).

Now consider BF theory. Let A be a $\mathtt{G}$-Lie algebra valued connection and let B be a End($\mathtt{G}$) field of $n-2$ forms on an n dimensional manifold X. The action in BF theory is then $S = \int_X \mathrm{tr}\,(BF_A)$, where F_A is the curvature of A. The equations of motion are given by varying B: $F_A = 0$, and varying A: $D_A B = 0$, where $D_A = d + A$.

Holomorphic BF theory (see Popov [7]) is similiar: On an n dimensional complex manifold X, let A be a $\mathtt{G}$-Lie algebra valued connection and let B be a End($\mathtt{G}$) field of $n, n-2$ forms. The action is then $S = \int_X \mathrm{tr}\,(BF_A^{0,2})$ where $F_A^{0,2}$ is the 0, 2 part of F_A. The equations of motion are given by varying B: $F_A^{0,2} = 0$ and varying A: $D_A^{0,1}B = 0$, where $D_A^{0,1} = \bar{\partial} + A^{0,1}$ is the 0, 1 part of D_A. Again, the first equation is the integrability condition for a holomorphic vector bundle. This is crucial for the Penrose-Ward transform.

For holomorhic BF theory on a complex supermaifold, X, the setup is similiar but B will be a section of $\mathtt{G} \otimes E_{n,n-2}$. The action for holomorphic BF theory on a supermanifold takes the form: $S = \int \mathrm{tr}\,(B \vee F_A^{0,2})$, where $\vee$ is contraction of sections of $\bigwedge^{0,2} TX$ with sections of $\bigwedge^{0,2} TX^*$. The equations of motion are as before: $F_A^{0,2} = 0$ and

$$\bar{\partial}' B + B \vee A^{0,1} - A^{0,1} \vee B = 0 \quad .$$

Again, the first equation is the condition for a holomorphic bundle. In several of the actions above, the path integral is over all 0, 1 connections which give almost complex structures on the bundles. We are thus integrating over a space of almost complex bundles with holomorphic bundles as extrema of the action.

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