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# WEIGHTED NORM BOUNDS FOR A LOCAL HÖLDER NORM OF ELLIPTIC AND OF PARABOLIC FUNCTIONS ON A NON-SMOOTH DOMAIN IN EUCLIDEAN SPACE

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**Abstract:** The rate of change of $u$, a solution to $Lu = \operatorname{div}\overrightarrow{f}$ in a bounded, nonsmooth domain $\Omega$, $u = 0$ on $\partial\Omega$, is investigated using a local Hölder norm of $u$ and different measures on $\Omega$. Results for both $L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x)\frac{\partial}{\partial x_j})$ and for $L = \partial/\partial t - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial}{\partial x_j})$ are presented.

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## 1. Introduction

I am going to discuss conditions on two Borel measures that are sufficient to give norm inequalities for a local Hölder norm of the solution $u(x)$ to the partial differential equation, $Lu(x) = \operatorname{div}\overrightarrow{f}(x)$, $u(x') = 0$ for $x' \in \partial\Omega$, with respect to the function $\overrightarrow{f}$. Here $L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{i,j}(x)\frac{\partial}{\partial x_j})$ is a strictly elliptic second order operator in divergence form with coefficients $a_{i,j}(x)$ bounded and measurable. $\Omega$ is a bounded domain in $\mathbb{R}^d$, whose boundary satisfies an exterior cone condition. So $\Omega$ could be a Lipschitz domain. The goal is to find conditions on $\mu$ and $\nu$, Borel measures defined on $\Omega$, so

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$$\left(\int_{\Omega}\left(\|u\|_{H^{\alpha}}(x)\right)^{q} d\mu(x)\right)^{1/q} \leq C\left(\int_{\Omega}\left(\left|\overrightarrow{f}(x)\right|^{p}+\left|div\overrightarrow{f}(x)\right|^{p}\right) d\nu(x)\right)^{1/p} \quad (1)$$

is valid for as large a range of $p$ and $q$ as possible. $C$ is independent of $u$, $\overrightarrow{f}$, $\mu$, $\nu$ and $q$. However, if $C$ is allowed to depend on $\mu(\Omega)$, $\nu(\Omega)$ and $q$, one can extend the range of indices for which (1) holds. After presenting this result and the Littlewood-Paley type inequality which also must be proved, a norm inequality for solutions to the parabolic equation,

$$(\frac{\partial}{\partial t}-L)u(x,t)=\operatorname{div}\overrightarrow{f}(x,t), u(x')=0 \text{ for } x'\in\partial_p\Omega_T,$$

will be stated. Although the inequality in the parabolic setting is similar to the inequality for an elliptic operator solution, the condition on the measures is quite different, involving a singular convolution instead of the local dyadic condition given in Theorem A.

The history of this problem started with harmonic functions on classical domains such as the unit disk and the upper half space, $\mathbb{R}^{d+1}_{+}$. In other words, the function $u(x)$ was a solution to the Dirichlet problem $\Delta u = 0$ in $\Omega$, with prescribed boundary data, $g(x')$. In 1985 Wheeden and Wilson [19] established necessary and sufficient conditions for two measures, $\mu$ on $\mathbb{R}^{d+1}_{+}$ and $\nu dx$ a measure defined on $\partial\Omega$, to have

$$\left(\int_{\mathbb{R}^{d+1}_{+}}\left(|\nabla u(x)|\right)^{q} d\mu(x)\right)^{1/q} \leq C\left(\int_{\mathbb{R}^{d}}\left|g(x')\right|^{p}\nu(x')dx'\right)^{1/p}$$

if $1 < p \leq q < \infty$ and $q \geq 2$. Here the boundary measure is given as a non-negative weight $\nu(x')$ multiplying Lebesgue surface measure.

Prior to that time Luecking [10], Shirokov [11], Verbitsky [16] and Videnskii [17] had studied versions of this problem on classical domains such as the unit disk and the upper half space. In fact for $\Omega = \mathbb{R}^{d+1}_{+}$, and $\nu(x') \equiv 1$, the measures $\mu$ have been completely characterized for $0 < p, q < \infty$. The approach of Wheeden and Wilson was to employ a dual operator argument which in turn required the application of a Littlewood-Paley inequality. Later work by the author and Wilson investigated the possibility of proving norm inequalities with $dx'$ replaced by harmonic measure on $\partial\Omega$, when $\Omega$ was taken to be a domain with a rough boundary – say a Lipschitz domain in $\mathbb{R}^d$. One major difficulty here was to obtain the Littlewood-Paley inequality for such a domain with reduced smoothness for the requisite function family $\{\varphi_{(Q)}\}_{Q\in\mathcal{D}}$. Overcoming this difficulty was made possible by work of Wilson [18] in which he proved a more general kind of square function inequality, involving the discreet analogue of the g* function of classical Littlewood-Paley theory, for functions of the form $\sum_{Q\in\mathcal{F}} \lambda_Q \varphi_{(Q)}(x')$ on $\mathbb{R}^d$. The $\lambda_Q \in \mathbb{R}^1$ are coefficients and the $\varphi_{(Q)}$ are taken from a family of functions that are indexed by dyadic cubes $Q$, and that satisfy certain decay, smoothness and cancellation conditions. The author and Wilson extended their results for harmonic $u$ to solutions of the elliptic and of the parabolic Dirichlet problem on non-smooth domains.

In these situations the functions $\varphi_{(Q)}$ that arise naturally are defined in terms of a kernel function that is associated with the differential operator on the domain $\Omega$. Estimates for the kernel functions such as geometric decay and Hölder continuity at the boundary have been proved by Cafarelli, Fabes, Mortola and Salsa [1], Jerison and Kenig [6] for elliptic operators, and by Fabes, Garofalo and Salsa [3], Fabes and Safonov [4], and by Kaj Nystrom [9] for solutions to $\left(\frac{\partial}{\partial t} - L\right) u(x,t) = 0$ on Lipschitz cylinders and Lipschitz $(1, \frac{1}{2})$ domains. Almost orthogonality can also be proved for functions of the form $f(x') = \sum_{Q\in\mathcal{F}} \lambda_Q \varphi_{(Q)}(x')$. Consequently it was possible to dominate the $L^p$ norm of such a function by the $L^p$ norm of a semi-discreet $g^*$ function, defined by

$$g^*(f)(x') = \left( \sum_{Q\in\mathcal{F}} \frac{\lambda_Q^2}{|Q|} \left( 1 + \frac{\left|x' - x'_Q\right|}{l(Q)} \right)^{-(d-\epsilon)} \right)^{1/2}.$$

Using the local Hölder norm

$$\|u\|_{H^\alpha}(x) \equiv \sup_{y\in P_{\delta(x)/50}} \frac{|u(x) - u(y)|}{|x-y|^\alpha},$$

with $\delta(x)$ = distance of $x$ from $\partial\Omega$ and $P_{\delta(x)/50} = \{w : |x - w| < \frac{\delta(x)}{50}\}$, it has been possible to find conditions on $\mu$ and $\nu$ sufficient to establish (1) for solutions to

$$Lu = \operatorname{div} \overrightarrow{f} \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$

I proved improved results for solutions to the Dirichlet problem for both elliptic and parabolic operators, when $|\nabla u|$ was replaced by $\|u\|_{H^\alpha}$. Last summer at this conference, I spoke about the theorem for $\left(\int_{\Omega_T} \|u(x,t)\|_{H^\alpha}^q \, d\mu(x,t)\right)^{1/q}$ for $\left(\frac{\partial}{\partial t} - L\right) u = 0$ in $\Omega_T$, a bounded domain in $\mathbb{R}^{d+1}$ with a time-varying lateral boundary.

For solutions to $Lu = \operatorname{div} \overrightarrow{f}$ in $\Omega$, $u = 0$ on $\partial\Omega$, the main result is Theorem

A. Assume that $d\nu$ is a measure on $\Omega$ that is mutually absolutely continuous with respect to Lebesgue measure. Let $d\sigma(x) = \left(\frac{d\nu}{dx}(x)\right)^{1-p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\mathcal{W}$ be a fixed collection of dyadic cubes in $\Omega$; each cube has the property that its dimension compares with its distance from $\partial\Omega$, and that a fixed dilate of the cube is again a Whitney-type cube wrt $\partial\Omega$. $\Omega \subset \bigcup_{Q\in\mathcal{W}} Q$. Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ whose boundary satisfies an exterior cone condition. For

$$M_1(Q_j) = l(Q_j)^{d/p'+1}\left(\frac{1}{|Q_j|}\int_{4Q_j}\left(\frac{d\nu}{dx}\right)^{s/(s-p)} dx\right)^{1/s-1/p},$$

$$M_2(Q_j) = \left(\int_\Omega \left(1 + \frac{|x - x_{Q_j}|}{l(Q_j)}\right)^{-(d-\epsilon)p'/2} d\sigma(x)\right)^{1/p'},$$

we have the following result.

**Theorem A.** *Suppose that for any $Q_j \in \mathcal{W}$*

$$\mu(Q_j)^{1/q}\cdot \max\left(M_1(Q_j); M_2(Q_j)\right) \le Cl(Q_j)^{d+\alpha}.$$

*Then there exists a constant $C$, independent of $u$, $\overrightarrow{f}$, $\mu$ and $\nu$ so that*

$$\left(\int_\Omega \left(\|u\|_{H^\alpha}(x)\right)^q d\mu(x)\right)^{1/q} \le C\left(\int_\Omega \left(\left|\overrightarrow{f}(x)\right|^p + \left|\mathrm{div}\,, \overrightarrow{f}(x)\right|^p\right) d\nu(x)\right)^{1/p} \quad (1)$$

*for $d < s < p \le q < \infty$.*

The proof of Theorem A is accomplished by applying Theorem B.

A measure $\sigma$ defined on a domain $D$, is said to be $A^\infty$ with respect to Lebesgue measure if for any cube $Q \subset D$ and any measurable subset $E$ of $Q$, there are fixed constants $C_0$ and $\kappa > 0$ so that $\left(\frac{\sigma(E)}{\sigma(Q)}\right)^\kappa \le C_0\frac{|E|}{|Q|}$, see [2].

**Theorem B.** *Let $Q_0 \supseteq \Omega$ be a large dyadic cube containing $\Omega$. Suppose that $\{\varphi_{(Q_j)}\}_{Q_j\in\mathcal{D}}$ is a family of functions defined on $Q_0$ and $= 0$ on $Q_0\backslash\Omega$, that satisfy conditions* **a**), **a′**), **b**) *and* **c**) *below. Let $h(x) = \sum_{J\in\mathcal{F}} \lambda_J\varphi_{(J)}(x)$, with $F$ a finite family of cubes from $W$. For any $p$, $0 < p < \infty$, there is a constant independent of $h$ so that for any measure $d\eta \in A^\infty(Q_0, dx)$, we have $\|h\|_{L^p(Q_0,d\eta)} \le C\,\|g^*(h)\|_{L^p(Q_0,d\eta)}$.*

The four conditions that hold for the family $\left\{\varphi_{(J)}(x)\right\}$ are:

$$\textbf{a)}\quad \left|\varphi_{(J)}(x)\right| \le Cl(J)^{2-d/2}\left(1+\frac{|x-x_J|}{l(J)}\right)^{2-d} \textit{ for all } x\in\Omega.$$

$$\textbf{a}')\quad \left|\varphi_{(J)}(x)\right| \le C\delta(x)^{\alpha} l(J)^{2-d/2-\alpha}\left(1+\frac{|x-x_J|}{l(J)}\right)^{2-d-\alpha} \textit{ for all } x\in\Omega.$$

$$\textbf{b)}\quad \left|\varphi_{(J)}(x)-\varphi_J(y)\right| \le C\left|x-y\right|^{\alpha} l(J)^{2-d/2-\alpha}\cdot \left(1+\frac{|x-x_J|}{l(J)}+\frac{|y-x_J|}{l(J)}\right)^{2-d-\alpha}$$

*for all $x$, $y$ in a Whitney-type region inside $4J$,* or $x, y\in\Omega\backslash 4J$.

$$\textbf{c)}\quad \int\left|\sum_{J\in\mathcal{F}}\lambda_J\varphi_{(J)}(x)\right|^2 dx \le C\sum_{J\in\mathcal{F}}\lambda_J^2.$$

Theorem B can be established by proving a "good-$\lambda$" inequality.

The definitions and background information needed to state the last result, Theorem C, can be found in [14].

**Theorem C.** *Suppose that $u(x,t)$ is a solution to*

$$(\partial/\partial t-\sum_{i,j=1}^{d}\frac{\partial}{\partial x_i}(a_{i,j}(x,t)\frac{\partial}{\partial x_j}))u(x,t)=\operatorname{div}\overrightarrow{f}(x,t) \text{ in } \Omega_T,\ u\big|_{\partial_p\Omega_T}=0$$

*with $\Omega_T$ as described above. Let $\mu$ and $\nu$ be Borel measures defined on $\Omega_T$ such that $\nu$ is absolutely continuous with Lebesgue measure. Let $d\sigma(x,t)=(\frac{d\nu}{dxdt}(x,t))^{1-p'}dxdt$. If there is a constant $C_0>0$ such that*

$$\mu(Q_j)^{1/q}\sup_{(x,t)\in\beta Q_j}\left(\int_{\Omega_T}\frac{d\sigma(y,s)}{\left(|x-y|+|t-s|^{1/2}\right)^{(d+\alpha)p'}}\right)^{1/p'}\le C_0\left|Q_j\right|^{1/q}$$

*for all cubes $Q_j\in\mathcal{W}$, then for all $0<q<\infty$ and $1<p<\infty$, there is a constant $C>0$, independent of $u$ and $\overrightarrow{f}$, so that*

$$\left(\int_{\Omega_T}\|u\|_{H^\alpha}^q(x,t)d\mu(x,t)\right)^{1/q}\le C\left(\int_{\Omega_T}\left|\operatorname{div}\overrightarrow{f}(y,s)\right|^p d\nu(y,s)\right)^{1/p}.$$

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