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NONLINEAR INTERACTION AND RESONANCE OF COUNTERPROPAGATING WAVES

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Abstract: A nonlinear partial differential equation with variable coefficients that describes wave motion in inhomogeneous material is considered. By assuming that variable material properties (the density and the elastic coefficient) have a weak variation the equation of motion is solved by making use of the perturbation method and the solution is sought in a series with a small parameter ε. The solution describes initial stage of nonlinear propagation, interaction and reflection of longitudinal waves in weakly inhomogeneous elastic material. The main features of wave interaction are illustrated by comparing solutions in homogeneous and inhomogeneous material, respectively. The analytical solution is studied numerically with the view to clarify the boundary oscillations in terms of the wave induced stress. Extensive numerical computations indicate that the influence of the parameters that describe weak inhomogeneous properties of the material on the amplitude-frequency dependence is close to linear.

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1. Introduction

Interaction of waves in different materials is under intensive investigation, see

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[4, 6, 11]. Wave-wave interaction is accompanied by the phenomenon of resonance [10]. Similarly to the resonance methods used in [5, 12] the wave-wave interaction resonance may be used for nondestructive characterization of materials.

The problem is studied in this paper. The counter-propagation of two longitudinal waves in weakly inhomogeneous nonlinear elastic materials [7, 8] is described analytically. The wave induced oscillations on the parallel boundaries of the material are analyzed. The effect of wave interaction resonance is clarified. The resonance is dependent on the inhomogeneous properties of the material, on the wave excitation amplitude and on the excitation frequency. Consequently, it may be used as a basis for algorithms of ultrasonic nondestructive evaluation of material properties.

2. Problem Formulation

The theory of elasticity with quadratic nonlinearity is characterized by Lamé constants λ and μ, by three third order constants of elasticity ν_1, ν_2, ν_3 and by the density ρ, see [1, 3]. The spatial inhomogeneity of the material induces a dependence of these constants on the coordinates. The one-dimensional equation of motion of the inhomogeneous material reads as [2]

$$[\,1 + k_1(X)\,U_{,X}(\,X,t)\,]\;U_{,XX}(\,X,t) + k_2(X)\,U_{,X}(\,X,t) + \;k_3(X)\,[U_{,X}(\,X,t)]^2 \\ - \,k_4(X)\,U_{,tt}(\,X,t) = 0\,, \quad (1)$$

where U is the displacement of material particle, X denotes the Lagrangian coordinate and t is the time. Functions with variables in the indices after the comma denote first or second order partial derivatives.

In the one-dimensional case five elastic coefficients group together as the linear coefficient of elasticity $\alpha(X)$ and the nonlinear coefficient of elasticity $\beta(X)$

$$\alpha(X) = \lambda(X) + 2\mu(X)\,, \quad \beta(X) = 2\;[\nu_1(X) + \nu_2(X) + \nu_3(X)]\;. \qquad (2)$$

Now, coefficients $k_i(X)$, $i = 1,\ldots,4$ in the equation of motion (1) may be expressed in terms of $\alpha(X)$, $\beta(X)$ and the variable density $\rho(X)$ in the form

$$k_0(X) = [\alpha(X)]^{-1}\;,\; k_1(X) = 3\,[\,1 + \,k_0(X)\,\beta(X)]\;,\; k_2(X) = k_0(X)\,\alpha_{,X}(X)\,, \\ k_3(X) = 3/2\,k_0(X)\,[\alpha_{,X}(X) + \beta_{,X}(X)]\;,\; k_4(X) = \rho(X)\,k_0(X)\,. \qquad (3)$$

Equation (1) with coefficients in equation (3) describes the one-dimensional nonlinear propagation of longitudinal waves in a physically nonlinear inhomogeneous elastic material.

Material with a weak spatial inhomogeneity is considered. Its properties $\rho(X)$, $\alpha(X)$ and $\beta(X)$ are represented in the common form

$$\gamma(X) = \gamma^{(1)} + \varepsilon\,\gamma^{(2)}(X)\,, \quad \gamma = \rho, \alpha, \beta\,, \tag{4}$$

where $\gamma^{(1)}$ depicts the main constant part of the material property and $\varepsilon\,\gamma^{(2)}(X)$, where $\varepsilon \ll 1$ is the weakly variable part of it. It is assumed that the weakly variable part is described by the third order polynomial

$$\gamma^{(2)}(X) = \gamma_{1\xi}X + \gamma_{2\xi}X^2 + \gamma_{3\xi}X^3\,,\ \xi = \rho,\ \alpha,\ \beta\,. \tag{5}$$

Functions (4) and (5) are introduced into equations (3) and (1). The resulting equation of motion (1) enables us to investigate simultaneous counter-propagation and interaction of two longitudinal waves with arbitrary and smooth initial profiles $\varphi(t)$ and $\psi(t)$ in the weakly inhomogeneous nonlinear elastic material. This wave propagation process is excited by the initial and boundary conditions

$$U(X,0) = U_{,t}(X,0) = 0\,,\ U_{,t}(0,t) = \varepsilon a_0 \varphi(t) H(t)\,,\ U_{,t}(L,t) = \varepsilon a_L \psi(t) H(t)\,, \tag{6}$$

where constants $\varepsilon\, a_0$ and $\varepsilon\, a_L$ determine the initial amplitudes of the waves, L notes the thickness of the material and $H(t)$ is the Heaviside function. Functions $\varphi(t)$ and $\psi(t)$ satisfy conditions $\max|\varphi(t)| = \max|\psi(t)| = 1$ and $\lim_{t\to 0}\varphi(t) = \lim_{t\to 0}\psi(t) = 0$.

3. Perturbative Solution

Wave propagation is investigated under the assumption that deformations of the material caused by wave motion are small but finite and the plastic deformations do not occur. Consequently, the considered problem has a natural small parameter – a small strain. This leads to the idea to solve the problem making use of the perturbation technique and to seek the solution to equation (1) in a series with a small parameter ε,

$$U(X,t) = \sum_{n=1}^{\infty} \varepsilon^n\, U^{(n)}(X,t)\,, \quad 0 < \varepsilon \ll 1. \tag{7}$$

In principle, small parameters in equations (4) and (6) may be of different order. The intention is to use nonlinear effects of wave propagation in nondestructive investigation of weakly inhomogeneous properties of the material. As shown in [9], the nonlinear effects of the wave motion contain maximum information about the variable properties of the material provided these parameters are of the same order. This case is considered henceforth.

Following the perturbation technique, the coefficients in equation (3) in consideration of equations (4) and (5) are expanded into Taylor series. For example, the coefficient $k_0(X)$ takes the form

$$k_0(X) = \frac{1}{\alpha^{(1)}}\left[1 - \varepsilon\frac{\alpha^{(2)}(X)}{\alpha^{(1)}} + \left(\varepsilon\frac{\alpha^{(2)}(X)}{\alpha^{(1)}}\right)^2 - \dots\right] . \tag{8}$$

The thus deduced polynomial coefficients up to the sixth order with respect to X, $k_j^{(i)}(X)$, $i = 1, \dots, 3$, $j = 1, \dots, 4$ and the series (7) are introduced into equation (1). Equating to zero terms of equal power in ε and neglecting terms higher than ε^3 the following set of equations is obtained:

$$\begin{aligned}
O(\varepsilon):\quad & U^{(1)}_{,XX}(X,t) - k_4^{(1)}\, U^{(1)}_{,tt}(X,t) = 0\,,\\
O(\varepsilon^2):\quad & U^{(2)}_{,XX}(X,t) - k_4^{(1)}\, U^{(2)}_{,tt}(X,t) = -k_1^{(1)}\, U^{(1)}_{,X}(X,t)\, U^{(1)}_{,XX}(X,t)\\
& -k_2^{(2)}(X)\, U^{(1)}_{,X}(X,t) + k_4^{(2)}(X)\, U^{(1)}_{,tt}(X,t)\,,\\
O(\varepsilon^3):\quad & U^{(3)}_{,XX}(X,t) - k_4^{(1)}\, U^{(3)}_{,tt}(X,t) = -k_2^{(2)}(X)\, U^{(2)}_{,X}(X,t)\\
& -k_2^{(3)}(X)\, U^{(1)}_{,X}(X,t) - k_3^{(2)}(X)\, \left[U^{(1)}_{,X}(X,t)\right]^2\\
& -k_1^{(1)}\, U^{(1)}_{,X}(X,t) U^{(2)}_{,XX}(X,t) - k_1^{(1)}\, U^{(1)}_{,XX}(X,t)\, U^{(2)}_{,X}(X,t)\\
& -k_1^{(2)}(X)\, U^{(1)}_{,XX}(X,t)\, U^{(1)}_{,X}(X,t) + k_4^{(2)}(X)\, U^{(2)}_{,tt}(X,t)\\
& +k_4^{(3)}(X)\, U^{(1)}_{,tt}(X,t).
\end{aligned} \tag{9}$$

Equations (9) permit to determine the first three terms in the series (7). The first term is the solution to the first equation in (9) under the initial and boundary conditions (6) where ε is neglected. It describes linear wave propagation in physically linear materials including the superposition of waves by interaction. In the time interval $0 \le t\,c\,/\,L \le 2$ it consists of four terms ($c^{-2} = k_4^{(1)}$)

$$\begin{aligned}
& U^{(1)}_{,t}(X,t) = a_0 H(\xi)\varphi(\xi) + a_L H(\eta)\psi(\eta) - a_0 H(\theta)\varphi(\theta) - a_L H(\zeta)\psi(\zeta)\,,\\
& \xi = t - \frac{X}{c}\,,\ \eta = t - \frac{L-X}{c}\,,\ \theta = t - \frac{2L-X}{c}\,,\ \zeta = t - \frac{X+L}{c}\,.
\end{aligned} \tag{10}$$

The first and second term in equation (10) describe waves exited at the boundaries $X = 0$ and $X = L$ that counter-propagate simultaneously into the material. The third and the forth term describe propagation of corresponding waves after reflection back to the boundaries of excitation.

The influence of the physical and geometrical nonlinearity and the material inhomogeneity is introduced into solution (7) by the second and subsequent terms. The second and the third term is the solution to the second and the third equation in (9) with known r.h.s. under the initial and boundary conditions equal to zero. Analytical expressions for these terms are derived making use of the Laplace integral transform that transforms the equations into second order ODEs with known r.h.s., which are solved directly. The final expressions for the terms $U^{(n)}(X,t), n = 2, 3$ are obtained after application of the inverse Laplace transform (p – transform parameter)

$$U^{(n)}(X,t) = \lim_{Y\to\infty} \frac{1}{2\,\pi\,i} \int_{\alpha-iY}^{\alpha+iY} e^{t\,p}\; U^{(n)\,\mathcal{L}}(X,p)\,dp\,. \tag{11}$$

The final analytical expressions for the second and the third term are too cumbersome to be presented here.

4. Sine Wave Interaction

With the view to study the problem of inhomogeneous elastic material characterization on the basis of wave interaction data, the initial profiles of the waves are defined by the sine function

$$\varphi(t) = \psi(t) = \sin(\omega t)\,, \tag{12}$$

where ω denotes the radial frequency.

The analytical solution equation (7) to describe simultaneous propagation and interaction of two harmonic waves with the same frequency and absolute value of initial amplitudes is derived making use of the software for symbolic computation *MAPLE*. The analytical expressions for the first three terms in the obtained solution are too cumbersome to report here and the solution is analyzed numerically.

As an illustration, relative nonlinear oscillations $U^{(2)}_{,X}$ on the boundaries $X = 0$ and $X = L$ of the material with thickness $L = 0.1$ m and the basic properties close to duraluminium ($\rho^{(1)} = 3000$ kg/m^3, $\alpha^{(1)} = 100$ GPa, $\beta^{(1)} =$ - 750 GPa) are plotted in Figure 1. The amplitude of oscillation in Figure 1 is relative to the excitation amplitude. Variation of material properties is described by the function $\alpha(X) = \alpha^{(1)} + \varepsilon\gamma_{2\alpha}\,X^2$, $\gamma_{2\alpha} = 25$ GPa/m^2. Two excited waves are characterized by the frequency $\omega = 1.03387$ x 10^6 rad/s, amplitudes $a_0 = -a_L = -c$ m/s and $\varepsilon = 10^{-4}$.

Nonlinear boundary oscillations in Figure 1 are characterized by double frequency with respect to the frequency of excitation and by amplification of the

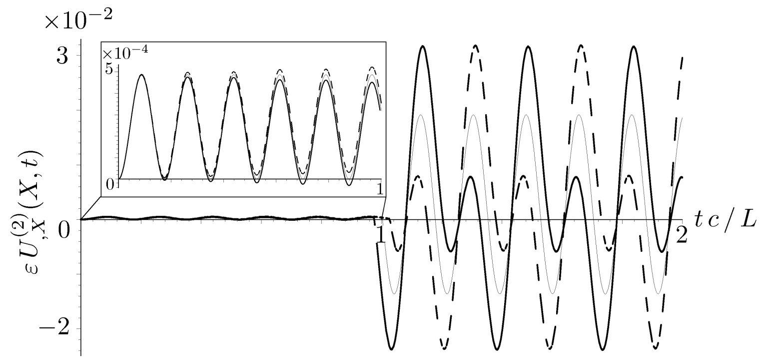

Figure 1: Influence of variable linear elastic properties on the nonlinear oscillations at the boundaries $X = 0$ (*bold solid line*) and $X = L$ (*dashed line*) of the material. *Solid line* - homogeneous material. The inset shows the solution in the interval $0 < t\,c/L < 1$ with amplified ordinate

oscillation amplitude in the interval of interaction $1 \leq t\,c\,/\,L < 2$. Oscillations in the interval of propagation $0 \leq t\,c\,/\,L < 1$ are zoomed in Figure 1 in separate plot. Essential is that nonlinear boundary oscillations in the homogeneous material have a constant amplitude in both intervals but the inhomogeneity of the material modulates them.

5. Nonlinear Resonance

Oscillations at the boundary $X = 0$ of the material described above are considered. Attention is focused to the first local maximum of the oscillation amplitude in the interaction interval $1 \leq t\,c\,/\,L < 2$. In the linear case, characterized by the first term $U^{(1)}_{,X}$ in solution (7) this is illustrated in Figure 2. The excitation frequency is chosen equal to $\omega_l = 2\,\pi\,n\,c/L\,,\;\; n = 3$ and in this case the interaction amplitude is three times higher than the amplitude of excitation, see [2]. The wave interaction is studied for 20 different excitation amplitudes chosen from the interval $0.35\,c \leq |a_0| \leq 1.3\,c$. The basic material properties are introduced in previous section. The resulting boundary oscillations for the first, sixth, eleventh and twentieth excitation amplitudes are plotted in Figure 2. The corresponding first local maxima of the amplitudes of the linear part of boundary oscillation are denoted by A_{lj}, $j = 1, 6, 11, 20$. The expres-

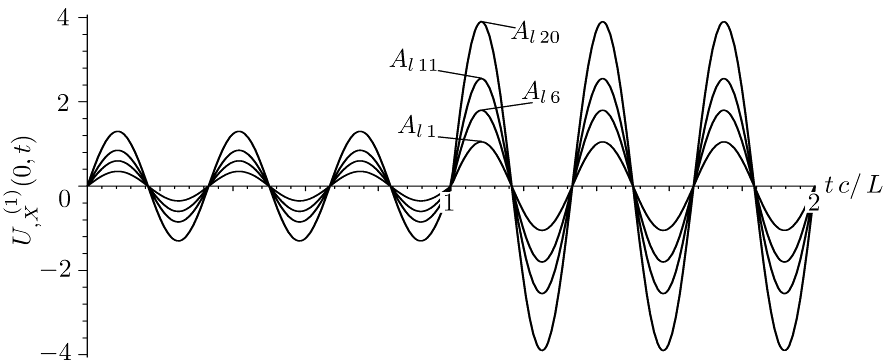

Figure 2: Linear part of oscillation at the boundary $X = 0$ caused by different excitation amplitudes

sion for the resonance frequency ω_l corresponds to the values of these maxima for the linear part and does not depend on nonlinearity and the inhomogeneous properties of the material.

The sensitivity of the values of the local maxima to the properties of material is caused by the higher order nonlinear effects of wave propagation and interaction, i.e., by the second and the subsequent terms in solution (7). Material inhomogeneity modulates the boundary oscillations (see Figure 1) and that is why the first positive peak (local maximum) of the interaction amplitude is considered. The value of this maximum, determined by the first three terms in (7) is sensitive to the spatial variation of the properties of material, to the wave excitation amplitude and frequency.

For convenience's sake the expression equation (4) that describes the variable properties of the material is transformed to

$$\gamma(X) = \gamma^{(1)} \left(1 + \delta_{1\xi}(X) + \delta_{2\xi}(X) + \delta_{3\xi}(X) \right)$$
$$\delta_{i\xi}(X) = \varepsilon\, \gamma_{1\xi} X^i / \gamma^{(1)}\,, \;\; i = 1, 2, 3,\; \gamma, \xi = \xi, \rho, \alpha, \beta. \tag{13}$$

Henceforth, the parameters of inhomogeneity are considered for a fixed value of material thickness $L = 0.1\,\mathrm{m}$ and $\delta_{i\xi} \equiv \delta_{i\xi}(L)$ characterizes a weak spatial variation of material properties.

The qualitatively different cases of inhomogeneous materials analyzed here are:

— a homogeneous material whose all nine parameters of inhomogeneity in equation (13) are simultaneously zero,

	case 1	case 2	case 3	case 4	case 5	case 6	case 7	case 8
$10^2\delta_{1\alpha}$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$10^2\delta_{2\alpha}$	1.0	1.0	1.0	1.0	-1.0	-1.0	-1.0	-1.0
$10^2\delta_{3\alpha}$	1.0	1.0	-1.0	-1.0	1.0	1.0	-1.0	-1.0
$10^2\delta_{1\rho}$	1.0	-1.0	1.0	-1.0	1.0	-1.0	1.0	-1.0

Table 1: Some examples of parameter values for more complex cases of inhomogeneity.

— a simple inhomogeneous material with only one non-zero parameter of inhomogeneity,

— a more complex case of inhomogeneity with four non-zero values of parameters of inhomogeneity represented in Table 1.

Figure 3 illustrates the numerical results for the first local maximum of boundary oscillation amplitude in the interaction interval. A is a numerically computed amplitude value and ω is a numerically computed resonance frequency value of nonlinear interaction resonance. Figure 3 depicts resonance amplitude cascades for six different materials. Here $\delta_{i\xi} = 0$ denotes the cascade of interaction resonance points in the case of homogeneous material. Cascades with $10^2\delta_{1\alpha}$, $10^2\delta_{2\alpha}$, $10^2\delta_{3\alpha}$, $10^2\delta_{1\rho}$ denote the results for four different inhomogeneous materials where only the indicated parameter is non-zero. Case 3 depicts the result for a more complicated case of inhomogeneous material where the non-zero parameters of inhomogeneity are given in Table 1.

It clears up from Figure 3 that the cascades corresponding to the inhomogeneous material are shifted with respect to the cascade for homogeneous material. Furthermore, the shifts of cascades for multi-parametric inhomogeneous materials with respect to cascade for the homogeneous case may be roughly determined as a vectorial sum of shifts that correspond to the one-parametric inhomogeneous materials. This is illustrated in Figure 3 by the arrows that connect the corresponding points in the cascades with the corresponding point in the cascade for homogeneous case.

Alike to Figure 3 results for the case 4 are given in Figure 4.

In comparison with the homogeneous material the weak variation of the nonlinear elastic coefficient $\beta(X)$ has a higher order small influence on the values of the resonance points than the variation of the linear elastic coefficient $\alpha(X)$ or the density $\rho(X)$.

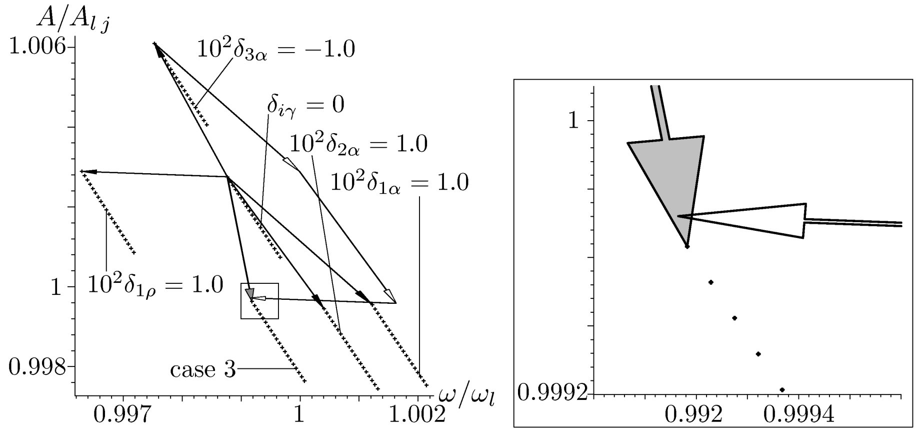

Figure 3: Influence of nonlinearity and physical inhomogeneity on the resonance frequency (case 3, $j = 1, \dots, 20$)

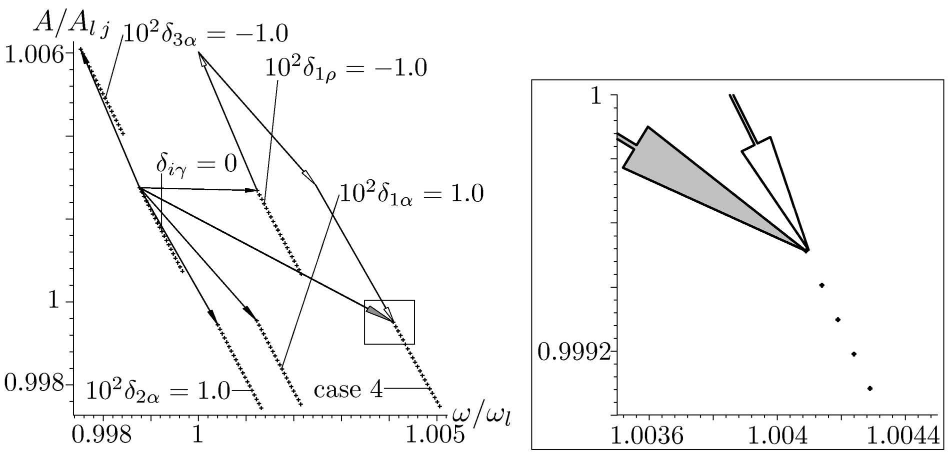

Figure 4: Influence of nonlinearity and physical inhomogeneity on the resonance frequency (case 4, $j = 1, \dots, 20$)

6. Conclusions

Simultaneous propagation of two longitudinal waves in a weakly inhomogeneous nonlinear elastic material is studied theoretically. The corresponding analytical solution is derived and analyzed numerically. Attention is paid to the oscillations on the boundaries of the material induced by wave interaction. The results of the analysis lead to the following conclusions:

— material inhomogeneity modulates boundary oscillations,

— the value of the boundary oscillation amplitude is sensitive to the material properties, wave excitation amplitude and frequency,

— numerically realized frequency scan determines the resonance value of the boundary oscillation amplitude,

— the value of the wave interaction resonance is function of material properties, wave excitation amplitude and frequency,

— the position of resonance cascades on the amplitude-frequency plane for the multi-parametric inhomogeneous material may be roughly determined as a sum of vectors that determine the position of cascades for the corresponding one-parametric inhomogeneous materials,

— the phenomenon of wave interaction resonance may be useful for algorithms of nondestructive evaluation (NDE) of material properties.

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