International Journal of Pure and Applied Mathematics
Volume 42 No. 2 2008, 217-225

SYMMETRIC α-STABLE SUBORDINATORS AND CAUCHY PROBLEMS

Erkan Nane
Department Statistics and Probability
Michigan State University
A413 Wells Hall, East Lansing, MI 48824, USA
e-mail: nane@stt.msu.edu
url: http://www.stt.msu.edu/∼nane

Abstract: We survey the results in Nane [24] and Baeumer, Meerschaert, and Nane [5] which deal with PDE connection of some iterated processes, and obtain a new probabilistic proof of the equivalence of the Higher order PDE's and fractional in time PDE's.

AMS Subject Classification: 60J65, 60K99
Key Words: iterated Brownian motion, PDE connection, $\alpha-$ stable process, α-time process, fractional diffusion, Lévy process, Cauchy problem, Brownian subordinator, Caputo derivative, fractional derivative in time

1. Introduction

In recent years, starting with the articles of Burdzy [9, 10], researchers had interest in iterated processes in which one changes the time parameter with one-dimensional Brownian motion.

To define *iterated Brownian motion* Z_t, due to Burdzy [9], started at $z \in \mathbb{R}$, let X_t^+, X_t^- and Y_t be three independent one-dimensional Brownian motions, all started at 0. *Two-sided Brownian motion* is defined to be

$$X_t = \begin{cases} X_t^+, & t \geq 0, \\ X_{(-t)}^-, & t < 0. \end{cases}$$

Then iterated Brownian motion started at $z \in \mathbb{R}$ is $Z_t = z + X(Y_t), \quad t \geq 0$.

Received: August 17, 2007

1.1. BM versus IBM

This process is not Markovian or Gaussian but it has many properties analogous to those of Brownian motion; we list a few.

Z_t has stationary (but not independent) increments, and is a *self-similar process* of index 1/4. *Laws of the iterated logarithm (LIL)* holds: usual LIL by Burdzy [9]

$$\limsup_{t\to\infty} \frac{Z(t)}{t^{1/4}(\log\log(1/t))^{3/4}} = \frac{2^{5/4}}{3^{3/4}}, \quad \text{a.s.}$$

Chung-type LIL by Khoshnevisan and Lewis [19] and Hu et al [17]. Khoshnevisan and Lewis [18] extended results of Burdzy [10], to develop a *stochastic calculus* for iterated Brownian motion. In 1998, Burdzy and Khosnevisan [12] showed that IBM can be used to model diffusion in a crack. Local times of this process was studied by Burdzy and Khosnevisan [11], Csáki, Csörgö, Földes, and Révész [13], Shi and Yor [29], Xiao [30], and Hu [16]. Bañuelos and DeBlassie [6] studied the *distribution of exit place* for iterated Brownian motion in cones. DeBlassie [14] studied the lifetime asymptotics of iterated Brownian motion in cones and Bounded domains. Nane [24, 22, 26, 27], extended some of the results of DeBlassie.

1.2. PDE Connection

In addition to the above properties of IBM there is an interesting connection between iterated Brownian motion and the *biharmonic operator* Δ^2; Allouba and Zheng [2] show that if we replace the outer process $X(t)$ in the definition of iterated Brownian motion with a continuous Markov process with generator L_x as the semigroup generator, then $u(t,x) = E_x[f(Z_t)] := E[f(Z_t)|Z_0 = x]$ solves the Cauchy initial value problem

$$\frac{\partial}{\partial t}u(t,x) = \frac{L_x f(x)}{\sqrt{\pi t}} + {L_x}^2 u(t,x); \quad u(0,x) = f(x),\ t>0, x\in\mathbb{R}^d. \tag{1.1}$$

When Z_t is an iterated Brownian motion, this was also obtained by DeBlassie [14] by a different method. Let $Z_t^1 = X(|Y_t|)$, then Allouba and Zheng [1] shows $E_x[f(Z_t)] = E_x[f(Z_t^1)]$.

1.3. Fractional Cauchy Problems

Zaslavsky (1994) [31] introduced the fractional kinetic equation

$$\frac{\partial^\beta}{\partial t^\beta} u(t,x) = L_x u(t,x); \quad u(0,x) = f(x) \tag{1.2}$$

for Hamiltonian chaos, where $0 < \beta < 1$ and L_x is the generator of some continuous Markov process $X_0(t)$ started at $x = 0$. Here $\partial^\beta g(t)/\partial t^\beta$ is the Caputo fractional derivative in time, which can be defined as the inverse Laplace transform of $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$, with $\tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt$ the usual Laplace transform.

Bajlekova [3] shows that the fractional Cauchy problem assumes unique solutions.

Baeumer and Meerschaert [4] and Meerschaert and Scheffler [21] show that the fractional Cauchy problem (1.2) is related to a certain class of subordinated stochastic processes; take D_t to be the stable subordinator, a Lévy process with strictly increasing sample paths such that $E[e^{-sD_t}] = e^{-ts^\beta}$, see for example Bertoin [8]. Define the inverse or hitting time or first passage time process

$$E_t = \inf\{x > 0 : D(x) > t\}. \tag{1.3}$$

The subordinated process $Z_t = X_0(E_t)$ occurs as the scaling limit of a continuous time random walk (also called a renewal reward process), in which iid random jumps are separated by iid positive waiting times (Meerschaert and Scheffler (2004), [21]). Theorem 3.1 in Baeumer and Meerschaert [4] shows that, in the case $p(t,x) = T(t)f(x)$ is a bounded continuous semigroup on a Banach space, the formula

$$u(t,x) = \int_0^\infty p((t/s)^\beta, x) g_\beta(s)\, ds = \frac{t}{\beta} \int_0^\infty p(x,s) g_\beta(\frac{t}{s^{1/\beta}}) s^{-1/\beta - 1} ds$$

yields a solution to the fractional Cauchy problem (1.2). Here $g_\beta(t)$ is the smooth density of the stable subordinator, with $\tilde{g}_\beta(s) = \int_0^\infty e^{-st} g_\beta(t)\, dt = e^{-s^\beta}$.

2. Brownian Subordiantors and Fractional Cauchy Problems

We give a probabilistic proof of the following theorem. A variation of this result was realized by Orsingher and Benghin [28] for a version of iterated Brownian motion.

Theorem 2.1. (Baeumer, Meerschaert, and Nane (2007), see [5]) *Let L_x be the generator of a Markov semigroup $T(t)f(x) = E_x[f(X_t)]$, and take $f \in D(L_x)$ the domain of the generator. Then, both the higher order Cauchy problem (1.1) and the fractional Cauchy problem (1.2) with $\beta = 1/2$, have the same solution*

$$u(t,x) \quad = \quad E_x[f(Z_t)] = \frac{2}{\sqrt{4\pi t}} \int_0^\infty T(s)f(x) \exp\left(-\frac{s^2}{4t}\right) ds. \tag{2.1}$$

Proof. E_t is the inverse of a $1-1/\alpha$ stable subordinator. E_t then is the local time of symmetric stable process of index α. In the case $\alpha = 2$, local time of Brownian motion is the same as $\sup_{0<s<t} B_s$. On the other hand, $\sup_{0<s<t} B_s$ and $|B_t|$ are same in distribution by the reflection principle. Hence E_t and $|B_t|$ have the same one-dimensional distributions, implying the result of the theorem. □

We obtain the following corollary of our theorem.

Corollary 2.2. (Baeumer, Meerschaert, and Nane (2007), see [5]) *For any continuous Markov process $X(t)$, both the Brownian-time subordinated process $X(|Y_t|)$ and the process $X(E_t)$ subordinated to the inverse $1/2$-stable subordinator have the same one-dimensional distributions. Hence they are both stochastic solutions to the fractional Cauchy problem (1.2), or equivalently, to the higher order Cauchy problem* (1.1).

In contrast to the previous, we have the following theorem.

Theorem 2.3. *Let Y be a symmetric stable process of index $1 < \alpha < 2$, and E_t is the inverse of a stable subordinator of index $1 - 1/\alpha$. The processes $X(E_t)$ and $X(|Y_t|)$ do not have same one-dimensional distribution.*

Proof. Let L_1^0 be the local time at $x = 0$. L_1^0 has the same one-dimenional distributions as E_t. Lemma 1 in Hawkes [15] implies that

$$P[L_1^0 > \lambda] \sim C_1 \lambda^{-\alpha/2} \exp(-C_{\alpha h} \lambda^\alpha).$$

Proposition 4 in Bertoin [8] says

$$P[Y_1 > u] \sim P[\sup_{0\le s\le 1} Y_s > u] \sim c u^{-\alpha}.$$

This tells us that in the case Y_t is a symmetric stable process of index $\alpha < 2$, $|Y_t|$ and E_t does not have same one-dimensional distributions. □

When the outer proces is Lévy process we have uniqueness of the solutions in Theorem 2.1. The proof relies on a Laplace-Fourier transform argument.

Theorem 2.4. (Baeumer, Meerschaert, and Nane (2007), see[5]) *Suppose that $X(t) = x + X_0(t)$, where $X_0(t)$ is a Lévy process starting at zero. If L_x is the generator of the semigroup $T(t)f(x) = E_x[(f(X_t))]$ on $L^1(\mathbb{R}^d)$, then for any $f \in D(L_x)$, both the initial value problem* (1.1), *and the fractional Cauchy problem* (1.2) *with $\beta = 1/2$, have the same unique solution given by* (2.1).

An easy extension of the argument for Theorem 2.4 shows that, under the same conditions, for any $n = 2, 3, 4, \ldots$ both the Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = \sum_{j=1}^{n-1} \frac{t^{1-j/n}}{\Gamma(j/n)} L_x^j f(x) + L_x^n u(t,x); u(0,x) = f(x) \tag{2.2}$$

and the fractional Cauchy problem (1.2) with $\beta = 1/n$ have the same unique solution given by $u(t,x) = \int_0^\infty p((t/s)^\beta, x) g_\beta(s)\, ds$ with $\beta = 1/n$. Hence the process $Z_t = X(E_t)$ is also the stochastic solution to this higher order Cauchy problem.

3. Other Subordinators

α-time process is a Markov process subordinated to the absolute value of an independent one-dimensional symmetric α-stable process: $Z_t = B(|S_t|)$, where B_t is a Markov process and S_t is an independent symmetric α-stable process both started at 0. Let $Z_t^x = x + Z_t$ the process started at x.

This process is self similar with index $1/2\alpha$ when the outer process X is a Brownian motion. In this case Nane [23] defined the Local time of this process and obtained Laws of the iterated logarithm for the local time for large time.

3.1. Pde-Connection

Theorem 3.1. (Nane (2005), see [25]) *Let $T(s)f(x) = E[f(X^x(s))]$ be the semigroup of the continuous Markov process $X^x(t)$ and let L_x be its generator. Let $\alpha = 1$. Let f be a bounded measurable function in the domain of L_x, with $D_{ij}f$ bounded and Hölder continuous for all $1 \le i,\ j \le n$. Then $u(t,x) = E[f(Z_t^x)]$ solves*

$$\frac{\partial^2}{\partial t^2} u(t,x) = -\frac{2L_x f(x)}{\pi t} - L_x^2 u(t,x);\ u(0,x) = f(x).$$

For $\alpha = l/m \neq 1$ rational: the PDE is more complicated since kernels of symmetric α-stable processes satisfy a higher order PDE:

$$(\frac{\partial^2}{\partial s^2})^l + (-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}})p_t^\alpha(0,s) = 0.$$

We also have to assume that we can take the operator out of the integral. This is valid for $\alpha = 1/m$, $m = 2, 3, \cdots$ by a lemma in Nane [25].

Theorem 3.2. (Nane (2005), see [25]) *Let $\alpha \in (0,2)$ be rational $\alpha = l/m$, where l and m are relatively prime. Let $T_s f(x) = E[f(X^x(s))]$ be the semigroup of the continuous Markov process $X^x(t)$ and let L_x be its generator. Let f be a bounded measurable function in the domain of L_x, with $D^\gamma f$ bounded and Hölder continuous for all multi index γ such that $|\gamma| = 2l$. Then $u(t,x) = E[f(Z_t^x)]$ solves*

$$(-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}}u(t,x) = -2\sum_{i=1}^{l}\left(\frac{\partial^{2l-2i}}{\partial s^{2l-2i}}p_t^\alpha(0,s)|_{s=0}\right)L_x^{2i-1}f(x) \; - \; L_x^{2l}u(t,x);$$

$$u(0,x) = f(x).$$

For some other connections of Pde's and iterated processes see papers by Nane [25] and Allouba and Zheng [2], Allouba [1], Baeumer et al [5] and references therein.

4. Open Problems

Question 1. Looking at the governing PDE for subordinators other than Brownian motion, are there any fractional in time PDE which has the same solution as the higher order PDE?

Question 2. Are there PDE connections of the iterated processes in bounded domain as the PDE connection of Brownian motion in bounded domains?

Acknowledgements

Author thanks Professor Mark M. Meerschaert and Professor Yimin Xiao for their help and discussions on the results in this paper.

References

[1] H. Allouba, Brownian-time processes: The PDE connection and the corresponding Feynman-Kac formula, *Trans. Amer. Math. Soc.*, **354**, No. 11 (2002), 4627-4637.

[2] H. Allouba, W. Zheng, Brownian-time processes: The PDE connection and the half-derivative generator, *Ann. Prob.*, **29**, No. 2 (2001), 1780-1795.

[3] E.G. Bajlekova, *Fractional Evolution Equations in Banach Spaces*, Ph.D. Thesis, Eindhoven University of Technology (2001).

[4] B. Baeumer, M.M. Meerschaert, Stochastic solutions for fractional Cauchy problems, *Fractional Calculus Appl. Anal.*, **4** (2001), 481-500.

[5] B. Baeumer, M.M. Meerschaert, E. Nane, Brownian subordinators and fractional Cauchy problems (2007), To Appear.

[6] R. Bañuelos, R.D. DeBlassie, The exit distribution for iterated Brownian motion in cones, *Stochastic Processes and their Applications*, **116**, No. 1 (2006), 36-69.

[7] P. Becker-Kern, M.M. Meerschaert, H.P. Scheffler, Limit theorems for coupled continuous time random walks, *The Annals of Probability*, **32**, No. 1B (2004), 730-756.

[8] J. Bertoin, *Lévy Processes*, Cambridge University Press (1996).

[9] K. Burdzy, Some path properties of iterated Brownian motion, In: *Seminar on Stochastic Processes* (Ed-s: E. Çinlar, K.L. Chung, M.J. Sharpe), Birkhäuser, Boston (1993), 67-87.

[10] K. Burdzy, Variation of iterated Brownian motion, In: *Workshops and Conference on Measure-valued Processes, Stochastic Partial Differential Equations and Interacting Particle Systems* (Ed. D.A. Dawson), Amer. Math. Soc. Providence, RI (1994), 35-53.

[11] K. Burdzy, D. Khoshnevisan, The level set of iterated Brownian motion, In: *Séminarie de Probabilités XXIX* (Ed-s: J Azéma, M. Emery, P.-A. Meyer, M. Yor), Lecture Notes in Mathematics, **1613**, Springer, Berlin (1995), 231-236.

[12] K. Burdzy, D. Khoshnevisan, Brownian motion in a Brownian crack, *Ann. Appl. Probabl.*, **8**, No. 3 (1998), 708-748.

[13] E. Csáki, M. Csörgö, A. Földes, P. Révész, The local time of iterated Brownian motion, *J. Theoret. Probab.*, **9** (1996), 717-743.

[14] R.D. DeBlassie, Iterated Brownian motion in an open set, *Ann. Appl. Prob.*, **14**, No. 3 (2004), 1529-1558.

[15] J. Hawkes, A lower Lipschitz condition for stable subordinator, *Z. Wahrsch. Verw. Gebiete*, **17** (1971), 23-32.

[16] Y. Hu, Hausdorff and packing measures of the level sets of iterated Brownian motion, *J. Theoret. Probab.*, **12**, No. 2 (1999), 313-346.

[17] Y. Hu, D. Pierre-Loti-Viaud, Z. Shi, Laws of iterated logarithm for iterated Wiener proceses, *J. Theoret. Probabl.*, **8** (1995), 303-319.

[18] D. Khoshnevisan, T.M. Lewis, Stochastic calculus for Brownian motion in a Brownian fracture, *Ann. Applied Probabl.*, **9**, No. 3 (1999), 629-667.

[19] D. Khoshnevisan, T.M. Lewis, Chung's law of the iterated logarithm for iterated Brownian motion, *Ann. Inst. H. Poincaré Probab. Statist.*, **32**, No. 3 (1996), 349-359.

[20] M.M. Meerschaert, D.A. Benson, H.P. Scheffler, B. Baeumer, Stochastic solution of space-time fractional diffusion equations, *Phys. Rev. E*, **65** (2002), 1103-1106.

[21] M.M. Meerschaert, H.P. Scheffler, Limit theorems for continuous time random walks with infinite mean waiting times, *J. Applied Probab.*, **41**, No. 3 (2004), 623-638.

[22] E. Nane, Iterated Brownian motion in bounded domains in $\mathbb{R}^n$, *Stochastic Processes and Their Applications*, **116** (2006), 905-916.

[23] E. Nane, Laws of the iterated logarithm for α-time Brownian motion, *Electron. J. Probab.*, **11**, No. 18 (2006), 434-459.

[24] E. Nane, Iterated Brownian motion in parabola-shaped domains, *Potential Analysis*, **24** (2006), 105-123.

[25] E. Nane, Higher order PDE's and iterated processes, *Trans. American Math. Soc.*, To Appear.

[26] E. Nane, Isoperimetric-type inequalities for iterated Brownian motion in $\mathbb{R}^n$, *Stat. Probab. Lett.*, To Appear.

[27] E. Nane, Lifetime asymptotics of iterated Brownian motion in $\mathbb{R}^n$, *Esaim: PS*, **11** (2007), 147-160.

[28] E. Orsingher, L. Beghin, Time-fractional telegraph equations and telegraph processes with Brownian time, *Prob. Theory Rel. Fields*, **128** (2004), 141-160.

[29] Z. Shi, M. Yor, Integrability and lower limits of the local time of iterated Brownian motion, *Studia Sci. Math. Hungar.*, **33**, No. 1-3 (1997), 279-298.

[30] Y. Xiao, Local times and related properties of multi-dimensional iterated Brownian motion, *J. Theoret. Probab.*, **11**, No. 2 (1998), 383-408.

[31] G. Zaslavsky, Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion, *Phys. D*, **76** (1994), 110-122.