International Journal of Pure and Applied Mathematics
Volume 42 No. 3 2008, 377-382

# ON EMBEDDING ORDERED PARTIAL GROUPOIDS INTO PARTIALLY ORDERED SEMIGROUPS

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**Abstract:** Two sets of axioms are presented which guarantee an ordered partial groupoid's embedding into a strict partially ordered semigroup.

**AMS Subject Classification:** 08A55, 20M99
**Key Words:** ordered partial groupoids, partially ordered semigroups, partial algebras

## 1. Introduction

Research began some time ago on the general problem of embedding partial groupoids into semigroups (see [5], [6], and [8]) and on determining axioms which guarantee that such a partial algebra can be embedded into a semigroup (see [1], [3], and [7]). We take that research one step further and determine axioms which guarantee an ordered partial groupoid's embedding into a strict partially ordered semigroup. A general characterization of such an embedding is provided in [4], but not axioms guaranteeing an embedding.

## 2. Definitions and Axioms

**Definition 1.** $\mathcal{A} = (A, \succsim, P, \circ)$ is an ordered partial groupoid if:

Received: August 17, 2007 

(1) $(A, P, \circ)$ is a partial groupoid, i.e., $A$ is a nonempty set, $P \subseteq A \times A$ a nonempty set, $\circ : P \to A$ a binary operation, and

(2) $\succsim$ is an antisymmetric, complete (i.e., for all $a, b \in A$, $a \succsim b$ or $b \succsim a$), and transitive relation on $A$.

If, in addition, there exists $1 \in A$ such that:

(3) $(a, 1), (1, a), \in P$, $a \circ 1 = 1 \circ a = a$, and $a \succsim 1$ for every $a \in A$, then $\mathcal{A}$ is a positively ordered partial groupoid.

From this point, we let $(A, P, \circ)$ denote a partial groupoid only and let $\mathcal{A}$ denote a positively ordered partial groupoid only.

**Definition 2.** $\mathcal{S} = (S, \geq^*, *)$ is a strict partially ordered semigroup if:

(1) $\geq^*$ is an antisymmetric, reflexive, and transitive relation on $S$,

(2) $(S, *)$ is a semigroup, and

(3) the strict monotony laws are satisfied, i.e., if $x >^* y$ (i.e., $x \geq^* y$ and $x \neq y$), then $x * z >^* y * z$ and $z * x >^* z * y$.

We present two sets of axioms on $\mathcal{A}$ which guarantee that it is *embeddable* in some strict partially ordered semigroup $\mathcal{S}$; that is, we present axioms which guarantee that there exists a function $\varphi : A \to S$ such that: (i) $a \succsim b$ if and only if $\varphi(a) \geq^* \varphi(b)$ and (ii) $\varphi(a \circ b) = \varphi(a) * \varphi(b)$.

We use letters $a$, $b$, $c$, $d$, $f$, and $g$ to denote elements of $A$ only and letters $i$, $j$, $m$, and $n$ to denote positive integers only. The *free semigroup* on $A$, denoted $(F_A, \cdot)$, is the set $\{a_1 \ldots a_n : a_i \in A,\ i = 1, \ldots, n\}$, where $a_1 \ldots a_n$ is a *word* in $F_A$, along with the binary operation $\cdot$ defined as $a_1 \ldots a_n \cdot b_1 \ldots b_m = a_1 \ldots a_n b_1 \ldots b_m$. We use the letters $u$, $v$, $w$, $x$, $y$, and $z$ to denote members of $F_A$ only. We call $w_i = uabv \to w_{i+1} = u\,(a \circ b)\,v$ a *direct move* from $w_i$ to $w_{i+1}$. We also write $x \to y$ (with the arrow from $x$ to $y$ in any direction) if $x = y$, or if there exist $w_i$, $i = 1, \ldots, n$, such that $x \to w_1 \to \cdots \to w_n \to y$, i.e., $y$ is arrived at from $x$ through a finite sequence of direct moves. We may also write $\pi_j\,(a_1 \ldots a_n) = b$ when $a_1 \ldots a_n \to b$, where the subscript $j$ denotes a particular positioning of parentheses in a product formed by $a_1, \ldots, a_n$, in that order.

Consider the following axioms, which might be satisfied by a partial groupoid $(A, P, \circ)$.

**Axiom 3.** (A, Associativity) *(1) If $\pi_1(a_1 \ldots a_n)$ and $\pi_2(a_1 \ldots a_n)$ are defined in $(A, P, \circ)$, they are equal.*

*(2) For all $(a, b), (c, d) \in P$ such that $a \circ b = c \circ d$, there exists $f \in A$ such that $a = c \circ f$ and $f \circ b = d$, or there exists $g \in A$ such that $a \circ g = c$ and*

$g \circ d = b$.

*(3) If $(a,b), (b,c) \in P$, where $b \neq 1$, then $(a \circ b, c)$, $(a, b \circ c) \in P$ and $(a \circ b) \circ c = a \circ (b \circ c)$.*

Given $\mathcal{A} = (A, \succsim, P, \circ)$ and $a_1 \ldots a_n, b_1 \ldots b_n \in F_A$ we also use $\succsim$ to write $a_1 \ldots a_n \succsim (\succ) b_1 \ldots b_n$ if $a_i \succsim b_i$ for $i = 1, \ldots, n$ (with $a_i \succ b_i$ for at least one $i$); notice that $\succsim$ on $F_A$ is antisymmetric, reflexive, and transitive. The following axioms might be satisfied by $\mathcal{A}$.

**Axiom 4.** (M, Monotonicity) *If $\pi_1(a_1 \ldots a_n)$ and $\pi_2(b_1 \ldots b_n)$ are defined in $(A, P, \circ)$ and $a_1 \ldots a_n \succ b_1 \ldots b_n$, then $\pi_1(a_1 \ldots a_n) \succ \pi_2(b_1 \ldots b_n)$.*

**Axiom 5.** (D, Decomposition) *(1) If $a \succ b$, $(b', b'') \in P$, and $b = b' \circ b''$, there exists $(a', a'') \in P$ such that $(a', a'') \succ (b', b'')$ and $a = a' \circ a''$.*

*(2) If $a \succ b$, $(a', a'') \in P$, and $a = a' \circ a''$, there exists $(b', b'') \in P$, where $(a', a'') \succ (b', b'')$ and $b = b' \circ b''$.*

*(3) If $(a,b) \succ (c,d)$, $(a,b) \in P$, and $c, d \in A$, then $(c,d) \in P$.*

Axioms (A1) and (M) are necessarily satisfied by a positively ordered partial groupoid which can be embedded in a strict partially ordered semigroup. Given a partial groupoid $(A, P, \circ)$, let $\kappa$ denote the congruence relation on $(F_A, \cdot)$ generated by the defining relations $(ab, a \circ b) \in \kappa$. Denote the congruence classes of $(F_A/\kappa, \cdot)$ by $\boldsymbol{x}$, where $x \equiv a_1 \ldots a_n$.

## 3. Preliminary Results

Proofs of the lemmas are in [2] or the references given there.

**Lemma 6.** *If $(A, P, \circ)$ satisfies (A1) and (A2), then it is embeddable into the semigroup $(F_A/\kappa, \cdot)$. Furthermore, if $x, y \in F_A$, where $x \kappa y$, there exists $z \in F_A$ such that $z \to x$ and $z \to y$.*

**Lemma 7.** *Let $x \succ y$, where $x, y \in F_A$.*

*(1) If $\mathcal{A}$ satisfies (D1) and $y' \to y$, where $y' \in F_A$, then there exists $x' \in F_A$ such that $x' \to x$ and $x' \succ y'$.*

*(2) If $\mathcal{A}$ satisfies (D2) and $x' \to x$, where $x' \in F_A$, then there exists $y' \in F_A$ such that $y' \to y$ and $x' \succ y'$.*

**Lemma 8.** *If $\mathcal{A}$ satisfies (M) and $x \succ y$, where $x, y \in F_A$, then not*

$$\begin{array}{ccc} x & \rightarrow & z, \\ & \nearrow & \\ y & & \end{array}$$

*where $z \in F_A$.*

**Lemma 9.** *Suppose that $\mathcal{A}$ satisfies (M). If there exists $z \in F_A$ such that*

$$\begin{array}{ccc} z & \rightarrow & xw \\ & \searrow & \\ & & yw \end{array} \quad \textit{or} \quad \begin{array}{ccc} z & \rightarrow & wx \\ & \searrow & \\ & & wy, \end{array}$$

*where $w, x, y, z \in F_A$, then there exists $v \in F_A$ such that*

$$\begin{array}{ccc} v & \rightarrow & x \\ & \searrow & \\ & & y. \end{array}$$

## 4. Main Results

**Theorem 10.** *If $\mathcal{A}$ satisfies (A1), (A2), (M), (D1), and (D2), then $\mathcal{A}$ is embeddable in a strict partially ordered semigroup.*

*Proof.* By Lemma 6, $(A, P, \circ)$ is embeddable in $(F_A/\kappa, \cdot)$. For $\boldsymbol{x}, \boldsymbol{y} \in F_A/\kappa$, define $\boldsymbol{x} \geq \boldsymbol{y}$ if there exist $\bar{x}, \bar{y} \in F_A$ such that $\bar{x} \rightarrow x$, $\bar{y} \rightarrow y$, and $\bar{x} \succsim \bar{y}$. It can be shown that the ordering is independent of the choices of the representatives of the congruence classes. Clearly, $\geq$ is reflexive and the weak monotony laws hold.

Suppose that $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{y} \geq \boldsymbol{x}$. By the definition of $\geq$, there exist $\bar{x}, x', \bar{y}, y' \in F_A$ such that

$$\begin{array}{ccc} \bar{x} & \rightarrow & x \\ & \nearrow & \\ x' & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \bar{y} & \rightarrow & y, \\ & \nearrow & \\ y' & & \end{array} \tag{1}$$

where

$$\bar{x} \succsim \bar{y} \text{ and } y' \succsim x'. \tag{2}$$

Given (1), by Lemma 6, there exists $\hat{x} \in F_A$ such that

$$\begin{array}{ccccc} \hat{x} & \rightarrow & \bar{x} & \rightarrow & x. \\ & \searrow & & \nearrow & \\ & & x' & & \end{array} \tag{3}$$

Given (1)-(3), by Lemma 7, there exist $\hat{y}$ and $y''$ such that

$$\begin{array}{ccccc} \hat{y} & \to & \bar{y} & \to & y, \\ & & & \nearrow & \\ y'' & \to & y' & & \end{array} \tag{4}$$

where

$$y'' \succsim \hat{x} \succsim \hat{y}. \tag{5}$$

Now if either inequality at (5) is an equality, then $x\kappa y$, which implies that $\boldsymbol{x} = \boldsymbol{y}$, and we are done showing antisymmetry. Otherwise, by the transitivity of $\succsim$, (5) implies that $y'' \succ \hat{y}$. But the latter inequality is impossible by Lemma 8, given (4).

Assume that $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{y} \geq \boldsymbol{z}$. Then there exist $\bar{x}, \bar{y}, y', z' \in F_A$ such that

$$\bar{x} \to x, \quad \begin{array}{ccc} \bar{y} & \to & y, \\ & \nearrow & \\ y' & & \end{array} \quad \text{and } z' \to z, \tag{6}$$

where

$$\bar{x} \succsim \bar{y} \text{ and } y' \succsim z'. \tag{7}$$

Given (6), by Lemma 6, there exists $\hat{y} \in F_A$ such that

$$\begin{array}{ccccc} \hat{y} & \to & \bar{y} & \to & y. \\ & \searrow & & \nearrow & \\ & & y' & & \end{array} \tag{8}$$

Given (6)-(8), by Lemma 7, there exist $x', \bar{z} \in F_A$ such that

$$x' \to \bar{x} \to x \text{ and } \bar{z} \to z' \to z, \tag{9}$$

where

$$x' \succsim \hat{y} \succsim \bar{z}. \tag{10}$$

Given (10), by the transitivity of $\succsim$, $x' \geq \bar{z}$. Then, given (9), by the definition of $\geq$, $\boldsymbol{x} \geq \boldsymbol{z}$.

The fact that $(F_A/\kappa, \cdot)$ is cancellative (i.e., $\boldsymbol{x}\cdot\boldsymbol{z} = \boldsymbol{y}\cdot\boldsymbol{z}$ or $\boldsymbol{z}\cdot\boldsymbol{x} = \boldsymbol{z}\cdot\boldsymbol{y}$ implies that $\boldsymbol{x} = \boldsymbol{y}$) follows from Lemmas 6 and 9. It follows almost immediately that since $(F_A/\kappa, \geq, \cdot)$ is a weak partially ordered semigroup which is cancellative, it is a strict partially ordered semigroup. □

The proof of the following theorem and its supporting lemmas are in [2].

**Theorem 11.** *If $\mathcal{A}$ satisfies (A3), (M), (D2), and (D3), then it is embeddable in a strict partially ordered semigroup.*

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