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GENERALISED SYMMETRIES AND THE CONNECTION TO DIFFERENTIAL SEQUENCES

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Abstract: In ordinary differential equations differential sequences are the analogues of the hierarchies found for partial differential equations such as the KdV hierarchy. We initiate our study by considering the generalised symmetries of the potential Burgers' equation and show how these lead naturally to the Riccati sequence, the first two members of which are the Riccati equation and the Painlevé-Ince equation. The properties of both the Riccati and Emden-Fowler sequences are intriguing and the generalised Riccati sequence gives some insight to the uniqueness associated with the former.

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1. Introduction

The use of sequences of real numbers and functions is introduced and appre-

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ciated early in one's mathematical life. Differential equations also germinate sequences, where the formula to generate the next member of the sequence is a recursion operator. The idea was mysteriously initiated through the context of hierarchies of partial differential equations. A well-known paradigm of such an hierarchy is the KdV hierarchy with a recursion operator of the form

$$\mathrm{RO}_{\mathrm{KdV}} = D_x^2 + 4u + 2u_x D_x^{-1}. \tag{1}$$

The first two members of the hierarchy are

$$\begin{aligned} u_t + u_{xxx} + 6uu_x &= 0\,, \\ u_t + u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x &= 0. \end{aligned}$$

Indeed for each soliton equation one can produce an infinite sequence (hierarchy) of partial differential equations. Besides soliton equations are related to the six Painlevé equations (see [1]) and therefore one can transform those hierarchies via symmetry (similarity) reductions to sequences of ordinary differential equations with the Painlevé equations as first members, see [9]. The general extension to ordinary differential equations has only been established recently (see [7]) and is based on the classification of equations connected with recursion operators for linearisable $1+1$ evolution partial differential equations, see [6, 14]. Euler et al (see [7]) simply removed the dependence on t to move from nonlinear partial differential equations to ordinary differential equations. However, a more systematic method for the determination of recursion operators for ordinary differential equations is needed. The word hierarchy pertains, but was deliberately changed to sequence (see [3]) for obvious reasons of continuation and logic.

2. Generalised Symmetries

In this section we try to substantiate a rigid mathematical reasoning for the existence of differential sequences in ordinary differential equations. We find that there is an intimate connection between the elements of the Riccati sequence (see [3]) and the generalised symmetries of the potential Burgers' equation.

Consider the potential Burgers' equation

$$u_t = u_{xx} + u_x^2. \tag{2}$$

When dealing with generalised symmetries an initial assumption on the structure of the coefficient functions is necessary. We search for all third-order generalised symmetries, i.e. symmetries of the form

$$G = \xi(t,x,u,u_x,u_{xx},u_{xxx})\partial_t + \eta(t,x,u,u_x,u_{xx},u_{xxx})\partial_x + \zeta(t,x,u,u_x,u_{xx},u_{xxx})\partial_u.$$

The ten fundamental characteristics the linear, constant-coefficient combinations of which produce all third-order generalised symmetries are (see [13])

$$\begin{aligned}
V_0 &= 1\,,\\
V_1 &= u_x\,,\\
V_2 &= tu_x + \tfrac{1}{2}x\,,\\
V_3 &= u_{xx} + u_x^2\,,\\
V_4 &= t(u_{xx} + u_x^2) + \tfrac{1}{2}xu_x\,,\\
V_5 &= t^2(u_{xx} + u_x^2) + txu_x + \left(\tfrac{1}{2}t + \tfrac{1}{4}x^2\right)\,,\\
V_6 &= u_{xxx} + 3u_xu_{xx} + u_x^3\,,\\
V_7 &= t(u_{xxx} + 3u_xu_{xx} + u_x^3) + \tfrac{1}{2}x(u_{xx} + u_x^2)\,,\\
V_8 &= t^2(u_{xxx} + 3u_xu_{xx} + u_x^3) + tx(u_{xx} + u_x^2) + \left(\tfrac{1}{2}t + \tfrac{1}{4}x^2\right)u_x\,,\\
V_9 &= t^3(u_{xxx} + 3u_xu_{xx} + u_x^3) + \tfrac{3}{2}t^2x(u_{xx} + u_x^2) + \left(\tfrac{3}{2}t^2 + \tfrac{3}{4}tx^2\right)u_x\\
&\quad + \left(\tfrac{3}{4}tx + \tfrac{1}{8}x^3\right)\,,
\end{aligned} \tag{3}$$

plus the infinite family of characteristics

$$V_\rho = \rho(t,x)\mathrm{e}^{-u},$$

where ρ is a solution to the heat equation. The Lie bracket of the symmetries which correspond to the characteristics V_6 and V_7 gives a route to the determination of a fourth-order generalised symmetry which we denote as the tenth of the list, i.e.

$$V_{10} = u_{xxxx} + 4u_xu_{xxx} + 3u_{xx}^2 + 6u_x^2u_{xx} + u_x^4.$$

The process can be repeated indefinitely and that would result in an endless listing of generalised symmetries of all orders. It can be proven with the use of Fréchet derivatives (see [13]) that equation (2) possesses two recursion operators. The one of relevance to this paper is

$$\mathrm{RO}_{(2)} = D_x + u_x,$$

which is suggested by inspection of the characteristics V_0, V_1, V_3, V_6 and V_{10}. Note that the same procedure can be followed for other partial differential equations, for example the heat equation and the Korteweg-de Vries equation possibly leading to other interesting differential sequences, interesting with respect to their potential properties.

3. Differential Sequences

In order to study this subset of characteristics joined together with the existence of $\mathrm{RO}_{(2)}$ we perform a reduction to one lesser dimension by setting $u_x = u$ thus resulting to the so-called Riccati sequence (see [3]) (we descard V_0 and V_1)

$$
\begin{aligned}
u' + u^2 &= 0\,, && (4)\\
u'' + 3uu' + u^3 &= 0\,, && (5)\\
u''' + 4uu'' + 3u'^2 + 6u^2u' + u^4 &= 0\,, && (6)\\
u'''' + 5uu''' + 10u'u'' + 10u^2u'' + 15uu'^2 + 10u^3u' + u^5 &= 0\,, && (7)\\
&\vdots \\
(D+u)^n u &= 0\,, && (8)
\end{aligned}
$$

where $u = u(x)$ and $D = d/dx$. We recognise the first two members as the Riccati equation and the Painlevé-Ince equation.

The properties of the Riccati sequence are remarkable (see [3]) and, as it is shown in subsequent papers (see [12, 8]), not exceptional. The following propositions have been proven in (see [3]).

Proposition. *The general member of the Riccati Sequence possesses the symmetries*

$$
\begin{aligned}
\Gamma_1 &= \partial_x\,,\\
\Gamma_2 &= x\partial_x - u\partial_u\,,\\
\Gamma_3 &= x^2\partial_x + (n - 2xu)\partial_u\,,
\end{aligned}
$$

with the algebra $sl(2,R)$. For (4) there are an additional infinity symmetries (as a first-order ordinary differential equation) and for (5) an additional five (the maximal eight for a second-order ordinary differential equation).

The singularity analysis of each member of the sequence shows that every equation is integrable in the sense of Painlevé (expansion in a Laurent series, integrability in terms of functions almost everywhere analytic). Thence Andriopoulos et al (see [3]) were able to provide the general solution as is stated in the following proposition.

Proposition. *The general solution of the n-th member of the Riccati Sequence, $n \geq 1$, is given by*

$$
u_n = \frac{\left(\sum_{i=0}^{n} A_i x^i\right)'}{\sum_{i=0}^{n} A_i x^i}, \tag{9}
$$

where the A_i, $i = 0, n$, are constants of integration.

Note that the fine details of the singularity analysis and the resulting patterns in terms of the parameters of the standard analysis (leading-order behaviour parameters and resonances) are intriguing and themselves provide sufficient indication of the underlying theoretical interest in differential sequences. Having studied the symmetry and singularity analyses of all members of the Riccati sequence Andriopoulos et al (see [**?**]) were interested in the complete characterisation in terms of symmetries of all the equations. For more details concerning the initiation and further developments of the notion of complete symmetry groups consult, see [10, 2, 4].

Proposition. *The complete symmetry group of (8) is given by the* $(n+1)$ *symmetries*

$$\Delta_i = -\exp\left[-\int u\mathrm{d}x\right]\{x^{i-1}u-(i-1)x^{i-2}\}\partial_u,\ i=1,n+1.$$

The Riccati sequence is proven to exhibit surprisingly rich properties. A natural research direction would be to determine whether an assumption of the form $D+\lambda u$ for the recursion operator and a starting Riccati differential equation $u'+\kappa u^2$ would result in a similar explosion of interesting results. As is shown explicitly in [11], $\lambda=\kappa$ is the sole requirement for the ensuing sequence to possess the Painlevé property (or the weak Painlevé property) and therefore admit the basic prerequisite towards a possible integration in closed-form. However, no actual generalisation is achieved since the dependent variable can always be rescaled to lead to the Riccati sequence for the obvious $\lambda=\kappa=1$.

The first three members of the Emden-Fowler sequence are (see [12])

$$w''+\tfrac{3}{2}w^2=0\,, \qquad (10)$$

$$w^{(iv)}+5ww''+\tfrac{5}{2}w'^2+\tfrac{5}{2}w^3=0\,, \qquad (11)$$

$$w^{(vi)}+7ww^{(iv)}+14w'w'''+\tfrac{21}{2}w''^2+\tfrac{35}{2}w^2w''+\tfrac{35}{2}ww'^2+\tfrac{35}{8}w^4=0\,, \qquad (12)$$

and subsequent elements are obtained with the use of the integro-differential recursion operator

$$\mathrm{RO}_{\mathrm{EF}}=D^2+2w-w'D^{-1}. \qquad (13)$$

Although the somehow continued lack in Lie point symmetries (in this case we find only two, ∂_x and $x\partial_x-2w\partial_w$) Leach et al (see [12]) report the hidden integrability in terms of analytic functions for each of the members of the Emden-Fowler sequence with, once again, mysteriously attractive patterns for the various parameters of the analysis.

It is important to observe that in all instances already studied concerning ordinary differential equations (see [3, 8, 12, 11]) the resulting members of the sequences are autonomous scalar ordinary differential equations. This need not be the case as was noted by Joshi, see [9]. The recursion operator is (originated basically from (1))

$$\mathrm{RO}_{n+1}\{q\} = \int \left(D^3 + 4qD + 2q'\right) \mathrm{RO}_n\{q\}\mathrm{d}x, \quad \mathrm{RO}_1\{q\} = q, \tag{14}$$

with the Painlevé II sequence being in its most general expression

$$(D + 2y)\,\mathrm{RO}_n\{y' - y^2\} = xy + \alpha_n, \quad n \geq 1, \tag{15}$$

where all α_ns are constants. The first couple of members are

$$\begin{aligned} y'' - 2y^3 &= xy + \alpha_1\,, \\ y^{(4)} - 10yy'^2 - 10y^2y'' + 6y^5 &= xy + \alpha_2. \end{aligned} \tag{16}$$

Firstly we note that (16) is the second of the six Painlevé transcendents. Once again one faces a differential sequence where higher members have no other interest than sheer theoretical. This is of course the case for hierarchies to be found for partial differential equations as well. We lastly accentuate the fact that in the case of (15) the recursion operator has a completely different structure from, say, $\mathrm{RO}_{\mathrm{KdV}}$ and $\mathrm{RO}_{(2)}$ in the sense that now the recursion operator itself cannot be written in explicit form, rather only recursively.

4. Concluding Remarks

Recursion operators for partial differential equations have been a matter of research interest already for some decades. The KdV hierarchy is a noteworthy exemplar. The determination of all recursion operators that an equation admits is a well-established procedure, see [5, 13]. Nevertheless in the field of ordinary differential equations it still remains, as far as we know, an open problem. What are the conditions for an operator to be termed recursion when it comes to ordinary differential equations? Assuredly that would by no means indicate an underlying interest in the resulting sequences. An explanation of the reason for the remarkable properties of differential sequences would be a most welcome development.

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