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A LOWER ESTIMATE OF THE MOVEMENT FOR TWISTED TORAL DIFFEOMORPHISMS

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Abstract: As for a diffeomorphism of the 2-dimensional torus isotopic to the id and without periodic points, we give a lower estimate for the length of a vector contained in the rotation set when the Ruelle invariant is positive.

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1. Introduction

Let $f : T^2 \to T^2$ be a C^∞-diffeomorphism isotopic to id. We choose an isotopy f_t ($t \in [0,1]$) from the id to f ($f_0 = \text{id}$, $f_1 = f$). For any $t \in \mathbf{R}$, we define f_t by $f_t = f_{t-\lfloor t \rfloor} \circ f^{\lfloor t \rfloor}$. Let $\overline{f}_t : \mathbf{R}^2 \to \mathbf{R}^2$ ($t \in \mathbf{R}$) denote the lift of f_t satisfying $\overline{f}_0 = \text{id}$. Then $\overline{f}_t$ satisfies

$$\overline{f}_t(x + (m,n)) = \overline{f}_t(x) + (m,n) \text{ for } m,n \in \mathbf{Z} \text{ and } x \in \mathbf{R}^2.$$

We define $K_n(x)$ for $n \in \mathbf{Z}_+$ and $x \in \mathbf{R}^2$ by

$$K_n(x) = \text{maximal}\{\|(D\overline{f}_n)_y - (D\overline{f}_n)_x\|/\|x-y\|\,;\, x \neq y, y \in \mathbf{R}^2\},$$

and furthermore, $m(A)$ for a matrix A by

$$m(A) = \inf\{\|A(v)\|\,;\, \|v\| = 1\}.$$

Theorem 1. *Let f be a C^∞-diffeomorphism of T^2 isotopic to* id *and without periodic points. Suppose that the Ruelle invariant $\rho_\mu(f)$ is positive for some f-invariant measure μ. Let $N = [4\pi/\rho_\mu(f)] + 1$. Then there is x_n*

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$(n = N, N+1, \cdots)$ *such that*

$$\frac{m(Df^n_{x_n})^2}{2K_n(x_n)} \leqq \|\overline{f}_n(x_n) - x_n\|.$$

Corollary 2. *Under the same condition as Theorem 1, the rotation set contains a vector longer than*

$$\inf_{x\in\mathbf{R}^2}\inf_{n\geqq 1}\frac{1}{n}\frac{m(Df^n_x)^2}{2K_n(x)}.$$

2. Twist and Fixed Points

Let G denote $GL_+(2;\mathbf{R})$. We denote by $\widetilde{G}$ the universal covering of G. For an element A of $\widetilde{G}$ (i.e. $A : [0,1] \to G$, $A(0) = e$), the rotation number $\Theta(A)$ was defined as follows: For any $s \in [0,1]$, the matrix ${}^TA(s)A(s)$ is a positive symmetric matrix, where ${}^TA(s)$ denotes the transpose of $A(s)$. Let S_s denote its square root. Then $A(s)S_s^{-1}$ is an orthogonal matrix. The path $s \mapsto A(s)S_s^{-1}$ is regarded as an element of $\widetilde{SO}(2) = \{B : [0,1] \to SO(2), B(0) = e\}$. Let $R : \mathbf{R} \to SO(2)$ denote the natural identification with respect to the angle and let $\widetilde{R} : \mathbf{R} \to \widetilde{SO}(2)$ denote its lift satisfying that $\widetilde{R}(0)$ is the constant map $s \mapsto e$. Then we define the rotation number $\Theta(A)$ by $\widetilde{R}^{-1}(s \mapsto A(s)S_s^{-1})$. Since $\widetilde{R}(\Theta(A)) = (s \mapsto A(s)S_s^{-1})$, we have $R(\Theta(A)) = A(1)S_1^{-1}$. In particular, $R(\Theta(A))$ is given by $A(1)$.

In another point of view, $\Theta(A)$ can also be defined as follows: Let A_s $(s \in [0,1])$ denote the element of $\widetilde{G}$ defined by $A_s(u) = A(su)$ for $u \in [0,1]$. Then $A(s) = R(\Theta(A_s))S_s$ as above. Thus the path $s \mapsto \Theta(A_s) \in \mathbf{R} = \widetilde{SO}(2)$ $(s \in [0,1])$ is the lift of $s \mapsto A(s)S_s^{-1} \in SO(2)$ joining 0 with $\Theta(A)$.

For a point y of $\mathbf{R}^2$ and $t \in \mathbf{R}$, we define $g_{t,y} : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$g_{t,y}(x) = \overline{f}_t(x+y) - \overline{f}_t(y).$$

We define an element $G_{n,y}$ $(y \in \mathbf{R}^2, n \in \mathbf{Z})$ of $\widetilde{G}$ by

$$G_{n,y}(s) = (Dg_{ns,y})_0 (= (D\overline{f}_{ns})_y)$$

for $s \in [0,1]$. Let $\Theta_{n,y} = \Theta(G_{n,y})$ and $S_{n,y} = \sqrt{{}^TG_{n,y}(1)G_{n,y}(1)}$. Then we have

$$(D\overline{f}_n)_y = R(\Theta_{n,y})S_{n,y}. \tag{1}$$

In particular, $m(Df^n_y) = m(S_{n,y})$.

Since the fundamental group of the identity component of the group of C^∞ diffeomorphisms of T^2 is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ (represented by rational translations), the number $\Theta_{n,y}$ is independent of the choice of the isotopy f_t ($t \in [0,1]$) (see [2]).

Let μ be an f-invariant probability measure. Then $\lim_{k\to\infty} \Theta_{k,y}/k$ exists a.e. μ (see [3]), and the Ruelle invariant $\rho_\mu(f)$ is defined by

$$\rho_\mu(f) = \int_{T^2} \lim_{k\to\infty} \frac{\Theta_{k,y}}{k}\, d\mu.$$

Lemma 3. *For a positive integer n, $r \geqq 0$ and $\theta \in \mathbf{R}$, the following inequality holds*

$$\|g_{n,y}(re^{i\theta}) - (Dg_{n,y})_0(re^{i\theta})\| \leqq K_n(y)r^2/2\,.$$

Proof. Let ℓ denote the arc defined by $\ell(s) = rse^{i\theta}$ ($0 \leqq s \leqq 1$). Then we have

$$\begin{aligned}
&\|g_{n,y}(re^{i\theta}) - (Dg_{n,y})_0(re^{i\theta})\| \\
&\qquad \leqq \left\| \int_0^1 (Dg_{n,y})_{\ell(s)}(re^{i\theta})ds - (Dg_{n,y})_0(re^{i\theta}) \right\| \\
&\qquad \leqq r \int_0^1 \|(Dg_{n,y})_{\ell(s)}(e^{i\theta}) - (Dg_{n,y})_0(e^{i\theta})\| ds \\
&\qquad \leqq r \int_0^1 \|(D\overline{f}_n)_{\ell(s)+y}(e^{i\theta}) - (D\overline{f}_n)_y(e^{i\theta})\| ds \leqq K_n(y)r^2/2. \quad \square
\end{aligned}$$

Lemma 4. *If $\Theta_{n,y} = \pi/2$ and $r\, m(Df_y^n) > \|\overline{f}_n(y) - y\| + K_n(y)r^2/2$ for some $r > 0$ and y, then $\overline{f}_n$ has a fixed point.*

Proof. For any θ, we have

$$\begin{aligned}
&|\langle (D\overline{f}_n)_y(re^{i\theta}) - re^{i\theta}, e^{i(\theta+\pi/2)}\rangle| = r|\langle (D\overline{f}_n)_y(e^{i\theta}), e^{i(\theta+\pi/2)}\rangle| \\
&= r|\langle R(\pi/2)S_{n,y}(e^{i\theta}), R(\pi/2)e^{i\theta}\rangle| \\
&= r|\langle S_{n,y}(e^{i\theta}), e^{i\theta}\rangle| \\
&\geqq r\, m(S_{n,y}) \quad \text{because } S_{n,y} \text{ is symmetric} \\
&= r\, m(Df_y^n).
\end{aligned}$$

By Lemma 3,

$$\begin{aligned}
&\|(\overline{f}_n(y + re^{i\theta}) - (y + re^{i\theta})) - ((D\overline{f}_n)_y(re^{i\theta}) - re^{i\theta})\| \\
&\quad = \|(\overline{f}_n(y) - y) + (\overline{f}_n(y + re^{i\theta}) - \overline{f}_n(y) - (D\overline{f}_n)_y(re^{i\theta}))\| \\
&\qquad \leqq \|\overline{f}_n(y) - y\| + K_n(y)r^2/2 < r\, m(Df_y^n).
\end{aligned}$$

Since $|\langle (D\overline{f}_n)_y(re^{i\theta}) - re^{i\theta}, e^{i(\theta+\pi/2)}\rangle|$ is the distance between $(D\overline{f}_n)_y(re^{i\theta}) - re^{i\theta}$ and the line parallel to $e^{i\theta}$ (Figure 1). Then $\overline{f}_n(y + re^{i\theta}) - (y + re^{i\theta})$ is

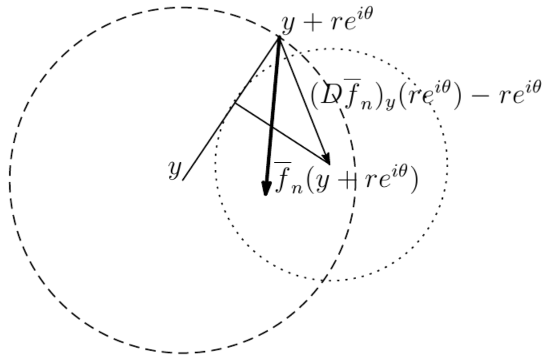

Figure 1: $X(x) = \overline{f}_n(x) - x$

not parallel to $e^{i\theta}$. We define a vector field X by $X(x) = \overline{f}_n(x) - x$. Let $D_r(y)$ denote the closed disc with center y and radius r. Since $X(y + re^{i\theta})$ is not parallel to $e^{i\theta}$, the index of X along $\partial D_r(y)$ is equal to 1. Thus there exists a singularity of X in $\operatorname{int} D_r(y)$, which is a fixed point of $\overline{f}_n$. □

3. Proof of the Main Theorem

Lemma 5. *If $\overline{f}_n$ has no fixed point and there is a point $y \in \mathbf{R}^2$ such that $\Theta_{n,y} = \pi/2$, then*

$$\frac{m(Df^n_y)^2}{2K_n(y)} \leqq \|\overline{f}_n(y) - y\|.$$

Proof. First observe that $K_n(y)$ is not equal to 0. In fact, if $K_n(y) = 0$, then $(D\overline{f}_n)_x$ is constant with respect to x. Now $\Theta_{n,y} = \pi/2$. Thus $D\overline{f}_n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} S_{n,y}$, and hence the trace of $D\overline{f}_n$ is necessary to be 0. This contradicts that f^n is isotopic to id. Therefore $K_n(y) \neq 0$.

Since $\overline{f}_n$ has no fixed point and $\Theta_{n,y} = \pi/2$,

$$r(m(Df^n_y) - K_n(y)r/2) \leqq \|\overline{f}_n(y) - y\|$$

for any $r > 0$. We put $r = m(Df^n_y)/K_n(y)$. Then we have $m(Df^n_y)^2/(2K_n(y)) \leqq \|\overline{f}_n(y) - y\|$. □

Proof of Theorem 1. Since $\rho_\mu(f)$ is positive, there is a point y_0 such that

$$\lim_{k\to\infty} \frac{\Theta_{k,y_0}}{k} > \frac{\rho_\mu(f)}{2}.$$

Let n be an integer greater than or equal to N. We claim that there is $N_1 \in \mathbf{Z}_+$ such that

$$\Theta_{N_1+n,y_0} - \Theta_{N_1,y_0} > 2\pi. \tag{2}$$

In fact, if not,

$$\frac{\Theta_{mn,y_0}}{mn} \leqq \frac{(\Theta_{mn,y_0} - \Theta_{(m-1)n,y_0}) + \cdots + (\Theta_{n,y_0} - \Theta_{0,y_0})}{mn} \leqq \frac{2\pi m}{mn} = \frac{2\pi}{n}.$$

Thus $\lim_{k\to\infty} \frac{\Theta_{k,y_0}}{k} \leqq \frac{2\pi}{n}$, which contradicts the assumption $4\pi/\rho_\mu(f) < N \leqq n$. Therefore, there is $N_1 \in \mathbf{Z}_+$ such that $\Theta_{N_1+n,y_0} - \Theta_{N_1,y_0} > 2\pi$.

Here we recall the property of $\Theta_{n,y}$ given in the appendix of [3]. Let $y_1 = \overline{f}_{N_1}(y_0)$. Then we have $(Dg_{n,y_1})_0(Dg_{N_1,y_0})_0 = (Dg_{N_1+n,y_0})_0$. Thus

$$\begin{aligned} &R(\Theta_{N_1+n,y_0})^{-1}R(\Theta_{n,y_1})R(\Theta_{N_1,y_0}) \\ &\qquad = S_{N_1+n,y_0}\left(S_{N_1,y_0}\right)^{-1}\left(R(\Theta_{N_1,y_0})^{-1}S_{n,y_1}R(\Theta_{N_1,y_0})\right)^{-1}. \end{aligned}$$

For any positive symmetric matrix, the inner product $\langle v, Sv\rangle$ is positive for any non-zero vector v. This concludes that $|\Theta_{N_1+n,y_0} - \Theta_{n,\overline{f}_{N_1}(y_0)} - \Theta_{N_1,y_0}| < 3\pi/2$. By (2), we have

$$\Theta_{n,\overline{f}_{N_1}(y_0)} > \frac{\pi}{2}. \tag{3}$$

We define a map $f_*^n : T^2 \times S^1 \to T^2 \times S^1$ ($S^1 = \mathbf{R}/2\pi\mathbf{Z}$) by

$$f_*^n(x,v) = \left(f^n(x), \frac{(D\overline{f}_n)_x(v)}{\|(D\overline{f}_n)_x(v)\|}\right).$$

Let $\widetilde{f}_*^n : T^2\times\mathbf{R} \to T^2\times\mathbf{R}$ denote the lift of f_*^n naturally defined by the isotopy f_t from the id. Then a map $\Delta f^n : T^2 \times S^1 \to \mathbf{R}$ is defined by

$$\widetilde{f}_*^n(x,\widetilde{v}) = (f^n(x), \widetilde{v} + \Delta f^n(x,v)),$$

where $\widetilde{v} \in \mathbf{R}$ is a lift of $v \in S^1$. In general, we have $|\Delta f^n(x,v) - \Theta_{n,x}| < \pi/2$ because

$$e^{i(\Delta f^n(x,v)+v)} = \frac{(D\overline{f}_n)_x(v)}{\|(D\overline{f}_n)_x(v)\|} = R(\Theta_{n,x})\frac{S_{n,x}(v)}{\|S_{n,x}(v)\|}.$$

By [2], there exists $(y_1, v_1) \in T^2 \times S^1$ such that $\Delta f^n(y_1,v_1) = 0$, and thus

$$|\Theta_{n,y_1}| < \pi/2. \tag{4}$$

For any $n \geqq N$, there is x_n such that $\Theta_{n,x_n} = \pi/2$ by (3) and (4). By

Lemma 5,

$$\frac{m(Df^n_{x_n})^2}{2K_n(x_n)} \leqq \|\overline{f}_n(x_n) - x_n\|. \qquad \square$$

The *rotation set* of f is the set of vectors

$$\lim_{i\to\infty} \frac{\overline{f}_{n_i}(x_i) - x_i}{n_i}$$

for some $x_i \in \mathbf{R}^2$ and $n_i > 0$ with $\lim_{i\to\infty} n_i = \infty$. Let $\{x_n\}_{n=N,N+1,\cdots}$ be a sequence obtained by Theorem 1. Now the set $\{\frac{\overline{f}_n(x_n)-x_n}{n}\}$ is bounded. By choosing a subsequence $\{n_i\}_{i=1,2,\cdots}$, we obtain a vector $\lim_{i\to\infty} \frac{\overline{f}_{n_i}(x_{n_i})-x_{n_i}}{n_i}$ in the rotation set, which satisfies the conclusion of Corollary 2.

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