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OPTIMAL DESIGN OF LMSE FIR DIGITAL FILTERS BY NUMERICAL OPTIMIZATION OF GENERALIZED RAYLEIGH QUOTIENT

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Abstract: Direct minimization of the mean square error at the output of an FIR digital filter operating stationary signal and noise is considered. It is shown that the problem can be re-stated as a problem of minimizing a GRQ. The latter is solved by a numerical line search algorithm that requires calculations of only two outer matrix products per iteration. Design results are compared with those obtained by using conventional approaches. Other applications of the procedure are outlined.

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1. Introduction

Alternative current operation of linear time invariant electronic systems is often considered using spectral domain. This is because these systems comply with the superposition principle, and sine waves are their eigenfunctions. Electronic filters are employed for enhancing particular spectral components (the signal) whilst attenuating some others (the noise). Many electronic signals and noises are considered as stationary random processes in the time domain, and so can be defined by their power spectra W_s and W_n respectively in the frequency domain. N. Wiener introduced the least mean square error (LMSE) criterion

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for the optimization of the transfer function of electronic filters

$$MSE = \int_0^\infty H^2 W_n \mathrm{d}\omega + \int_0^\infty (1-H)^2 W_s \mathrm{d}\omega \rightarrow \min, \quad (1)$$

where H is the transfer function of the filter and ω is the angular frequency, see [10]. The first term quantifies the power of noise penetrating to the output of the filter, and the second term gives the power of distortions to the signal at frequencies where the transfer function differs from unity. If there are no constraints on H then the optimal transfer function H_{opt} can be calculated at every frequency independently by [1]

$$H_{opt} = \frac{W_s}{W_s + W_n}. \quad (2)$$

Realizable devices impose various constraints on their transfer function. Some of these devices can be designed in a view to approximate the ideal Wiener filter (2). However some other electronic filters could be optimized by using the LMSE criterion (1) directly.

Finite impulse response (FIR) digital filters with positive symmetry and even number of coefficients provide the following transfer function

$$H(\omega) = exp\left(-j\omega\frac{N-1}{2}\Delta t\right)\sum_{k=1}^{\frac{N}{2}} a_k cos\left[\left(k-\frac{1}{2}\right)\omega\Delta t\right], \quad (3)$$

where $a_1 \dots a_{\frac{N}{2}}$ are independent coefficients representing samples of the device's impulse response, N is the number of these samples, Δt is the sampling interval, see [2]. The exponent describes the frequency response of the delay introduced by the filter. In most applications some delay is allowable thus this term could be excluded from the further analysis, and the model reduces to a linear combination of *cos* base functions:

$$H(\omega) \propto \sum_{k=1}^{\frac{N}{2}} a_k \varphi_k(\omega) \quad \in \quad \mathbf{R}. \quad (4)$$

Substitution of (4) into (1) yields

$$\begin{aligned} MSE \quad \propto \quad & \int_0^\infty \left(\sum_k a_k\varphi_k\right)^2 W_n \mathrm{d}\omega + \int_0^\infty \left(1 - \sum_k a_k\varphi_k\right)^2 W_s \mathrm{d}\omega \\ & = A^T M_{sn} A - 2B^T A + P_s \rightarrow \min, \quad (5) \end{aligned}$$

where

$$m_{sn}(k,l) = \int_0^\infty (W_s + W_n)\varphi_k\varphi_l \mathrm{d}\omega;$$
$$b(k) = \int_0^\infty W_s\varphi_k \mathrm{d}\omega; \quad P_s = \int_0^\infty W_s \mathrm{d}\omega. \tag{6}$$

Differentiation of equation (5) with respect to A gives the following optimal values for the vector of coefficients A_{opt}

$$A_{opt} = M_{sn}^{-1}B \tag{7}$$

that provide

$$MSE_{min} = P_s - B^T M_{sn}^{-1} B. \tag{8}$$

Numerical determination of A_{opt} is complicated in practice by a frequent ill-conditioned nature of M_{sn}, and a large number of coefficients. Also it is difficult to assess how well an approximate solution of equation (7) optimizes (5) thus complicating the quantization of the coefficients required for realizable devices. This is why an alternative optimization approach was developed.

2. Substituting LMSE Optimization with GRQ Optimization

A change in the level of the output signal (e.g. sound volume) does not affect the output error in relation to the output signal. If the output of the filter is amplified g times then the MSE becomes

$$MSE = g^2 A^T M_{sn} A - 2gB^T A + P_s \to \min. \tag{9}$$

Differentiation with respect to g yields the following optimal value for the gain g_{opt} that can be found for any A:

$$g_{opt} = \frac{B^T A}{A^T M_{sn} A}. \tag{10}$$

Substitution of g_{opt} back into (9) results in a new target function TF for the optimization:

$$MSE = P_s - \frac{(B^T A)^2}{A^T M_{sn} A} \to \min \;\Rightarrow\; TF = \frac{(B^T A)^2}{A^T M_{sn} A} \to \max. \tag{11}$$

This function represents a generalized Rayleigh quotient (GRQ). It can be optimized in a way as above, by equating its gradient

$$\nabla TF = \nabla \left[\frac{(B^T A)^2}{A^T M_{sn} A} \right] = 2 \times TF \left(\frac{B}{B^T A} - \frac{M_{sn} A}{A^T M_{sn} A} \right) \tag{12}$$

to zero. This yields the following equation:

$$\frac{B^T A}{A^T M_{sn} A} M_{sn} A = B. \tag{13}$$

Direct substitution of A_{opt} from equation (7) satisfies equation (13) thus the derived criterion agrees with the original one. However equation (13) can not be solved directly as the vector of coefficients A is present in both the numerator and the denominator. Therefore the optimization was conducted numerically using the gradient descent instead.

An important advantage of criterion (11) over criterion (5) is its dimensionless that allows for clearer interpretation of the optimization results.

3. Numerical Gradient Optimization of the GRQ

Various optimization methods can be employed using the gradient of the target function determined above. The simplest of those is the line search algorithm that considers the line defined by the present point in the coefficients' space A and the direction of ∇TF, see [9]. In the general case the optimum point on this line could be found by a some form of trial-and-error approach that takes considerable computing resources. However in this case the point can be determined by finding a scalar value for λ directly. It holds

$$\begin{aligned} TF^{(n+1)} &= \frac{\left(B^T \times (A^{(n)} + \lambda G^{(n)})\right)^2}{(A^{(n)} + \lambda G^{(n)})^T M_{sn} (A^{(n)} + \lambda G^{(n)})} \\ &= \frac{\lambda^2 \left(B^T G\right)^2 + 2\lambda^1 \left(B^T G\right)\left(B^T A\right) + \lambda^0 \left(B^T A\right)^2}{\lambda^2 \left(G^T M_{sn} G\right) + 2\lambda^1 \left(G^T M_{sn} A\right) + \lambda^0 \left(A^T M_{sn} A\right)} \\ &= \frac{a_n \lambda^2 + b_n \lambda^1 + c_n \lambda^0}{a_d \lambda^2 + b_d \lambda^1 + c_d \lambda^0} \to \max . \end{aligned} \tag{14}$$

Differentiation of the target function with respect to λ yields the following equation

$$\begin{aligned} TF'_\lambda &= P_3(\lambda) = 0 \\ \lambda^3 &: \quad 2a_n a_d - 2a_d a_n = 0 \\ \lambda^2 &: \quad 2a_n b_d + b_n a_d - 2a_d b_n - b_d a_n = a_n b_d - a_d b_n \\ \lambda^1 &: \quad 2a_n c_d + b_n b_d - 2a_d c_n - b_n b_d = 2a_n c_d - 2a_d c_n \\ \lambda^0 &: \quad b_n c_d - b_d c_n \end{aligned} \tag{15}$$

that reduces to a quadratic equation. It is possible to prove algebraically that this equation always has a positive discriminant and therefore two real roots,

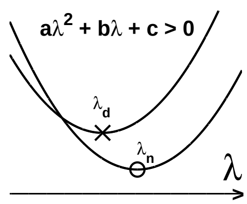

Figure 1: Sketch of numerator and denominator of TF

see [6]. However the following consideration shows it in an easier way. Both the numerator and denominator in fraction (14) represent some electrical powers that are always positive. Consequently TF could be represented as a ratio of two second order polynomials that behave as shown in Figure 1. The ratio will have its maximum close to the minimum of the denominator λ_d, and its minimum close to the minimum of the numerator λ_n. Hence equation (15) must always have two distinct real roots located outside $[\lambda_d; \lambda_n]$ if λ_d and λ_n differ. The root that is the closest to λ_d maximizes TF and should be used. The computational algorithm applies the gradient descent procedure iteratively until the change of A is smaller than desired:

1. Calculate M_{sn}; B; select initial A.
2. Calculate $C = M_{sn}A$; B^TA; C^TA.
3. Calculate G, equation (10).
4. Calculate $D = M_{sn}G$; B^TG; D^TA; D^TG.
5. Determine λ_{opt} from equation (13).
6. Update $A + \lambda_{opt}G \rightarrow A$.
7. If $\frac{\|\lambda_{opt}G\|}{\|A\|} < \epsilon$ stop, else *goto* 2.

Every iteration of the algorithm requires roughly $2N^2$ multiplications for calculations of C and D. After exiting the loop the coefficients are quantized, and the obtained MSE is compared with the optimal one. If the difference is significant, some extra iterations could be conducted. However this situation seems unlikely, and has not happened in practice so far.

4. Design Example (After [5])

The optimal frequency response can be approximated using the Fourier series. This method implicitly sets zero values of H_{opt} outside the interval where it is specified, and involves direct calculations without any need for optimization.

Some optimized approximations within a particular interval could be achieved by using *MATLAB* subroutines *remez.m* (for Chebyshev approximation) or *firls.m* (for least square approximation). Using *firls.m* for optimization invoked warnings on numerical accuracy (even for $\frac{N}{2} = 20$), and *remez.m* ceased to produce meaningful results for $\frac{N}{2} > 40$.

These alternatives were studied for comparison with the gradient GRQ optimization. Different initial points for the last alternative were applied. It was found that the procedure could converge to different maxima, and could not improve the results provided by either *remez.m* or *firls.m*. The quickest convergence (that also resulted in the highest value of the target function) was observed when the results of the Fourier approximation were used as the initial point. After less than 10 iterations the target function did not rise for more than 0.1%, and the relative change in A was found to be less than 10^{-5}.

Both *remez.m* and *firls.m* led to high values of the transfer function in the "do not care" regions that exceeded 50 dB, see [5]. These surges render both filters unusable in practice because they magnify any negligible noise (e.g., quantization noise) from these regions well above the signal.

5. Other Applications for Electronic Engineering (After [4])

The combination of two above procedures was found to be useful for various applied problems. In particular, other types of the FIR filters (see [2]) were optimized by using either LMSE or signal-to-noise ratio (see [8]) criteria in the frequency domain. FIR filters were also optimized in the time domain to produce a desired response on a particular input. This allowed for a uniform design of devices ranging from the matched filter to the inverse filter (see [3]).

The described approach was found applicable to the optimization of antenna arrays in both wide-band and pulse modes (see [4]).

This approach can account for random deviations of the coefficients that occur at the time of manufacturing of the optimized electronic device. These deviations could severily degrade, e.g., responses of surface acoustic wave devices (see [4]).

6. Conclusions

Optimization related to LMSE problems is often conducted by equating the gradient of the target function to zero, and solving the resulting system of linear equations. However the exact numerical solution could be difficult to calculate if the relevant matrix is ill-conditioned and/or the number of the coefficients is large. Some difficulties with this approach can be resolved by re-stating the original problem in terms of optimization of a generalized Rayleigh quotient.

The latter problem can be conveniently solved by a numerical gradient optimization. This is because the target function can be represented as a ratio of second-order polynomials on any line in the coefficients' space. Hence the optimal point for the line search can be found in one direct step per iteration.

This procedure was found to be useful, computationally fast and robust for a number of technical applications related to optimization of systems described by a weighted sum of basic functions using various energy criteria.

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