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A NONSTANDARD DIFFERENCE APPROACH FOR A 2-DIMENSIONAL SINGULAR PERTURBATION PROBLEM

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Abstract: We combine the two ideas of writing high order derivatives coming from Taylor series expansions in terms of the lower order derivatives for a two dimensional test problem, and rewriting the error of the expansions to reinforce diagonal dominance of the resulting system. We develop two techniques using nine equally spaced points and present numerical results.

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1. Introduction

In this paper we develop a method to solve the two dimensional singular perturbation boundary value problem

$$\begin{aligned} &-\epsilon u_{xx}(x,y) - \epsilon u_{yy}(x,y) + u_x(x,y) \\ &\qquad = 2\epsilon\pi^2 \sin(\pi x)\sin(\pi y) + \pi cos(\pi x)\sin(\pi y)\,, \qquad (1) \\ &u(0,y)=0, \quad u(1,y)=0, \quad u(x,0)=0, \quad u(x,1)=0. \end{aligned}$$

The exact solution of (1) is $u(x,y)=\sin(\pi x)\sin(\pi y)$.

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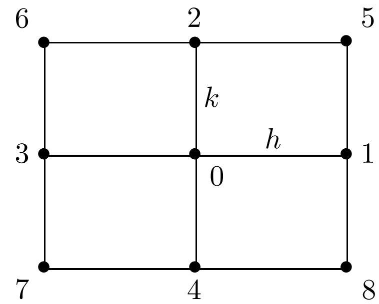

Figure 1: Nine points used for the two dimensional case

Dekema and Schultz [2] used this problem as a test problem. They used Taylor series expansions out to the fourth order derivatives and rewrote higher order derivatives in terms of the lower ones using the original partial differential equation in (1). They obtained remarkably good results for $\epsilon \geq 10^{-5}$, however due to instability, their technique did not give good results for smaller values of ϵ.

Segal [3] as well as Choo and Schultz [1] used a slightly different test problem for their two dimensional case. They also encountered stability problems for $\epsilon \leq 10^{-7}$.

We considered the next term in the Taylor series expansions and tried two different approximations to convert it to reinforce diagonal dominance and achieved stability.

2. Method

We start by dividing (1) by $-\epsilon$, and letting $\omega = 1/\epsilon$,

$$u_{xx}(x,y) + u_{yy}(x,y) - \omega u_x(x,y) + 2\pi^2 \sin(\pi x)\sin(\pi y) + \omega\pi\cos(\pi x)\sin(\pi y) = 0.$$

To develop our finite difference scheme, we divide the rectangular region bounded by $x = 0$, $y = 0$, $x = 1$ and $y = 1$ into $n \times m$ smaller rectangles where n and m are the number of subintervals along the x-axis, and y-axis respectively. We denote the step size along the x-axis by h, and the step size along the y-axis by k. We use nine points as in Figure 1.

We denote the point (x_i, y_j) as 0, (x_{i+1}, y_j) as 1, $u(x_i, y_{j+1})$ as 2,..., etc. Thus we will have $u(x_i, y_j) = u_0$, $u(x_{i+1}, y_j) = u_1$,...,$u(x_{i+1}, y_{j-1}) = u_8$. We set up our difference approximation as the following:

$$u_{xx}\mid_0 + u_{yy}\mid_0 - \omega u_x\mid_0 + 2\pi^2\sin(\pi x_0)\sin(\pi y_0) + \omega\pi\cos(\pi x_0)\sin(\pi y_0) \equiv \sum_{i=0}^{8}\alpha_i u_i + \alpha_9 = 0\,. \quad (2)$$

We put α_9 to collect terms which do not contain $u_i,\ i = 0,..,8$. Next we use the Taylor series expansions of $u_i,\ i = 1,..,8$ about the point x_0:

$$\begin{aligned}
&\sum_{i=0}^{8}\alpha_i u_i + \alpha_9 \approx \alpha_0 u_0\\
+\quad& \alpha_1\{u_0 + hu_x\mid_0 + \frac{h^2}{2}u_{xx}\mid_0 + \frac{h^3}{6}u_{xxx}\mid_0 + \frac{h^4}{24}u_{xxxx}\mid_0 + \frac{h^5}{120}u_{xxxxx} + \ldots\}\\
+\quad& \ldots + \alpha_8\{u_0 + hu_x\mid_0 - ku_y\mid_0 + \frac{1}{2}[h^2u_{xx}\mid_0 - 2hku_{xy}\mid_0 + k^2u_{yy}\mid_0]\\
+\quad& \frac{1}{6}[h^3u_{xxx}\mid_0 - 3h^2ku_{xxy}\mid_0 + 3hk^2u_{xyy}\mid_0 - k^3u_{yyy}\mid_0]\\
+\quad& \frac{1}{24}[h^4u_{xxxx}\mid_0 - 4h^3ku_{xxxy}\mid_0 + 6h^2k^2u_{xxyy}\mid_0 - 4hk^3u_{xyyy}\mid_0\\
+\quad& k^4u_{yyyy}\mid_0] + \frac{1}{120}h^5u_{xxxxx} + \ldots\} + \alpha_9. \qquad (3)
\end{aligned}$$

Notice that we also have the term $\frac{1}{120}h^5u_{xxxxx}$ as the only fifth order derivative term in the above expansion. We will use it as the converted error term in the future.

Next we write u_{xxx}, u_{xxxx} and u_{yyyy} in terms of lower order derivatives by differentiating the original equation with respect to x and y multiple times.

$$u_{xxx} = \omega u_{xx} - u_{xyy} - 2\pi^3\cos(\pi x)\sin(\pi y) + \pi^2\omega\sin(\pi x)\sin(\pi y), \qquad (4)$$

$$\begin{aligned}
u_{xxxx} &= \omega^2u_{xx} - \omega u_{xyy} - \omega\pi^3\cos(\pi x)\sin(\pi y) + \pi^2\omega^2\sin(\pi x)\sin(\pi y)\\
&- u_{xxyy} + 2\pi^4\sin(\pi x)\sin(\pi y), \qquad (5)
\end{aligned}$$

$$u_{yyyy} = -u_{xxyy} + \omega u_{xyy} + 2\pi^4\sin(\pi x)\sin(\pi y) + \pi^3\omega\cos(\pi x)\sin(\pi y). \qquad (6)$$

Once (4), (6) and (5) are substituted into (3) and terms are rearranged, the following is obtained:

$$\begin{aligned}
&\sum_{i=0}^{8}\alpha_i u_i + \alpha_9 \approx \{\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8\}u_0\\
+\quad& hu_x\mid_0\{\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8\} + ku_y\mid_0\{\alpha_2 - \alpha_4 + \alpha_5 + \alpha_6 - \alpha_7\\
-\quad& \alpha_8\} + u_{xx}\mid_0\{\frac{1}{2}h^2(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8) + \frac{1}{6}\omega h^3(\alpha_1 - \alpha_3 + \alpha_5
\end{aligned}$$

$$
\begin{aligned}
- &\quad \alpha_6 - \alpha_7 + \alpha_8) + \frac{1}{24}\omega^2 h^4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)\} + \frac{1}{2}k^2 u_{yy} \mid_0 \\
. &\quad \{\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8\} + hk u_{xy} \mid_0 \{\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8\} \\
+ &\quad \frac{1}{2}h^2 k u_{xxy} \mid_0 \{\alpha_5 + \alpha_6 - \alpha_7 - \alpha_8\} + u_{xyy} \mid_0 \{\frac{1}{2}hk^2(\alpha_5 - \alpha_6 - \alpha_7 + \alpha_8) \\
- &\quad \frac{1}{6}h^3(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8) - \frac{1}{24}\omega h^4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 \\
+ &\quad \alpha_8) + \frac{1}{24}\omega k^4(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)\} + \frac{1}{6}k^3 u_{yyy} \mid_0 \{\alpha_2 - \alpha_4 \\
+ &\quad \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8\} + \frac{1}{6}h^3 k u_{xxxy} \mid_0 \{\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8\} + u_{xxyy} \mid_0 \\
. &\quad \{-\frac{1}{24}h^4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8) + \frac{1}{4}h^2 k^2(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8) \\
- &\quad \frac{1}{24}k^4(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)\} + (\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8)\frac{1}{6}hk^3 u_{xyyy} \\
&\quad + \quad \{\alpha_9 + \frac{1}{6}h^3(-2\pi^3\cos(\pi x_0)\sin(\pi y_0) + \pi^2\omega\sin(\pi x_0)\sin(\pi y_0) \\
&\quad \cdot \quad (\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8) + \frac{1}{24}h^4(-\omega\pi^3\cos(\pi x_0)\sin(\pi y_0) \\
&\quad + \quad \pi^2\omega^2\sin(\pi x_0)\sin(\pi y_0) + \pi^4\sin(\pi x_0)\sin(\pi y_0)) \\
&\quad . \quad (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8) + \frac{1}{24}k^4(2\pi^4\sin(\pi x_0)\sin(\pi y_0) \\
&\quad + \quad \pi^3\omega\cos(\pi x_0)\sin(\pi y_0))(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)\} \\
&\quad + \quad \frac{1}{120}h^5 u_{xxxxx}(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8) + \ldots . \qquad (7)
\end{aligned}
$$

Setting the coefficients of (7) equal to the coefficients of the original partial differential equation and solving the system yield the following values of α's:

$$\alpha_1 = -2\alpha_5 + \frac{12 + 2\omega^2 h^2}{12h^2 + \omega^2 h^4} - \frac{\omega}{2h}, \quad \alpha_2 = \frac{1}{k^2} - \frac{1}{3k^2}\ \frac{6 + \omega^2 h^2}{12 + \omega^2 h^2} - \frac{1}{6h^2},$$

$$\alpha_3 = 2\alpha_6 + \frac{\omega}{2h} + \frac{12 + 2\omega^2 h^2}{12h^2 + \omega^2 h^4}, \quad \alpha_4 = \alpha_2,$$

$$\alpha_5 = \frac{1}{2h^2 k^2}[\frac{6 + \omega^2 h^2}{12 + \omega^2 h^2}\ \frac{\omega h^3 + 2h^2}{6} - \frac{\omega h^3}{6} + \frac{k^2}{6} - \frac{\omega h k^2}{12}],$$

$$\alpha_6 = -\alpha_5 + \frac{1}{h^2 k^2}[\frac{h^2}{3}\ \frac{6 + \omega^2 h^2}{12 + \omega^2 h^2} + \frac{k^2}{6}], \quad \alpha_7 = \alpha_6, \quad \alpha_8 = \alpha_5,$$

$$\alpha_9 = 2\pi^2\sin(\pi x_0)\sin(\pi y_0) + \pi\omega\cos(\pi x_0)\sin(\pi y_0) + \frac{\omega h^2}{6}[-2\pi^3\cos(\pi x_0)$$

$$\begin{aligned}
. \quad & \sin(\pi y_0) + \pi^2\omega\sin(\pi x_0)\sin(\pi y_0)] - \frac{k^2}{12}[2\pi^4\sin(\pi x_0)\sin(\pi y_0) \\
+ \quad & \pi^3\omega\cos(\pi x_0)\sin(\pi y_0)] - \frac{1}{24}h^4[-\omega\pi^3\cos(\pi x_0)\sin(\pi y_0) + \pi^2\omega^2\sin(\pi x_0) \\
. \quad & \sin(\pi y_0) + 2\pi^4\sin(\pi x_0)\sin(\pi y_0)]\frac{24+4\omega^2h^2}{12h^2+\omega^2h^4}. \qquad (8)
\end{aligned}$$

We now want to convert part of the next term in the Taylor series expansion,

$$\frac{1}{120}u_{xxxxx}(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8). \qquad (9)$$

We start by rewriting the fifth derivative u_{xxxxx} in terms of u_{xx}.

$$\begin{aligned}
& u_{xxxxx} = \omega^3u_{xx} - \omega^2u_{xyy} - \omega u_{xxyy} - u_{xxxyy} - \omega^2\pi^3\cos(\pi x)\sin(\pi y) \\
+ \quad & \omega^3\pi^2\sin(\pi x)\sin(\pi y) + \omega\pi^4\sin(\pi x)\sin(\pi y) + 2\pi^5\cos(\pi x)\sin(\pi y). \quad (10)
\end{aligned}$$

Note that $\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = -\omega/h$ by former calculations. Then we have

$$\frac{1}{120}h^5u_{xxxxx}\mid_0(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8) = -\frac{1}{120}\omega h^4u_{xxxxx}\mid_0 .$$

After substituting equation (10) for u_{xxxxx}, we choose to put the terms with u_{xyy}, u_{xxyy}, u_{xxxyy} back into the error and keep

$$\begin{aligned}
- \quad & \frac{\omega^4h^4}{120}u_{xx}\mid_0 + \frac{\omega^3h^4\pi^3}{120}\cos(\pi x_0)\sin(\pi y_0) - \frac{\omega^4h^4\pi^2}{120}\sin(\pi x_0) \\
. \quad & \sin(\pi y_0) - \frac{\omega^2h^4\pi^4}{120}\sin(\pi x_0)\sin(\pi y_0) - \frac{\omega h^4\pi^5}{60}\cos(\pi x_0)\sin(\pi y_0) \quad (11)
\end{aligned}$$

as the part of the error to be converted. When we use central difference approximations on $u_{xx}\mid_0$ and incorporate it into our scheme, only α_0, α_1, α_3 and α_9 are changed as in the following:

$$\alpha_0^* = \alpha_0 - \frac{1}{60}\omega^4h^2, \quad \alpha_1^* = \alpha_1 + \frac{1}{120}\omega^4h^2, \quad \alpha_3^* = \alpha_3 + \frac{1}{120}\omega^4h^2,$$

$$\begin{aligned}
\alpha_9^* \quad = \quad & \alpha_9 - \frac{1}{120}\omega h^4[\omega^2\pi^3\cos(\pi x_0)\sin(\pi y_0) - \omega^3\pi^2\sin(\pi x_0)\sin(\pi y_0) \\
- \quad & \omega\pi^4\sin(\pi x_0)\sin(\pi y_0) - 2\pi^5\cos(\pi x_0)\sin(\pi y_0)].
\end{aligned}$$

In Table 1, we give the results of this technique in terms of the maximum error. It shows that approximating u_{xx} by the three point central difference formula results in stability. In the next table (Table 2), we include Dekema and Schultz's [2] results for this problem. Although their results for relatively large ϵ values are remarkably good, their method does not converge for $\epsilon = 10^{-4}$ or $\epsilon = 10^{-5}$ when $n = 10$. Moreover, it is not stable when ϵ becomes smaller.

We tried a second technique where we incorporated the converted forms of

n	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-7}$
10	$1.2 \cdot 10^{-2}$	$8.3 \cdot 10^{-3}$	$8.3 \cdot 10^{-3}$	$8.3 \cdot 10^{-3}$	$8.3 \cdot 10^{-3}$	$8.3 \cdot 10^{-3}$
20	$3.4 \cdot 10^{-3}$	$2.3 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$
40	$2.8 \cdot 10^{-4}$	$9.8 \cdot 10^{-4}$	$5.2 \cdot 10^{-4}$	$5.1 \cdot 10^{-4}$	$5.1 \cdot 10^{-4}$	$5.1 \cdot 10^{-4}$

Table 1: Maximum error with n subdivisions in the x- and y-directions

n	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$
10	$3 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	$n.r.$	$n.r.$
20	$2 \cdot 10^{-5}$	$2 \cdot 10^{-5}$	$2 \cdot 10^{-5}$	$2 \cdot 10^{-5}$
40	$2 \cdot 10^{-6}$	$2 \cdot 10^{-6}$	$2 \cdot 10^{-6}$	$2 \cdot 10^{-6}$

Table 2: Maximum error in [6] with n subdivisions in the x- and y-directions

u_{xyy}, u_{xxyy} into our scheme. For $u_{xyy|_0}$, we considered

$$u_{xyy} \mid_0 = \frac{u_x \mid_2 - 2u_x \mid_0 + u_x \mid_4}{k^2}, \quad u_x \mid_2 = \frac{u_5 - u_6}{2h},$$
$$u_x \mid_0 = \frac{u_1 - u_0}{h}, \quad u_x \mid_4 = \frac{u_8 - u_7}{2h},$$

thus

$$u_{xyy} \mid_0 = \frac{u_5 - u_6 - 4u_1 + 4u_0 + u_8 - u_7}{2hk^2}. \tag{12}$$

For $u_{xxyy|_0}$ we used

$$u_{xxyy} \mid_0 = \frac{u_{xx} \mid_2 - 2u_{xx} \mid_0 + u_{xx} \mid_4}{k^2},$$

and using three point central difference approximations for $u_{xx} \mid_2$, $u_{xx} \mid_0$, and $u_{xx} \mid_4$ we obtained

$$u_{xxyy} \mid_0 = \frac{u_6 - 2u_2 + u_5 - 2u_3 + 4u_0 - 2u_1 + u_7 - 2u_4 + u_8}{h^2k^2}. \tag{13}$$

Substituting (12) and (13) in the error resulted in the following new α's:

$$\alpha_0^{**} = \alpha_0 - \frac{\omega^4h^2}{60} - \frac{\omega^3h^3}{60k^2} - \frac{\omega^2h^2}{30k^2}, \quad \alpha_1^{**} = \alpha_1 + \frac{\omega^4h^2}{120} + \frac{\omega^3h^3}{60k^2} + \frac{\omega^2h^2}{60k^2},$$
$$\alpha_2^{**} = \alpha_2 + \frac{\omega^2h^2}{60k^2}, \quad \alpha_3^{**} = \alpha_3 + \frac{\omega^4h^2}{120} + \frac{\omega^2h^2}{60k^2}, \quad \alpha_4^{**} = \alpha_2^{**},$$
$$\alpha_5^{**} = \alpha_5 - \frac{\omega^3h^3}{240k^2} - \frac{\omega^2h^2}{120k^2}, \quad \alpha_6^{**} = \alpha_6 + \frac{\omega^3h^3}{240k^2} - \frac{\omega^2h^2}{120k^2},$$

n	$\epsilon = 10^{-2}$	$\epsilon = 10^{-3}$	$\epsilon = 10^{-4}$	$\epsilon = 10^{-5}$	$\epsilon = 10^{-6}$	$\epsilon = 10^{-7}$
10	$7.9 \cdot 10^{-2}$	$1.7 \cdot 10^{-3}$	$7.3 \cdot 10^{-3}$	$8.2 \cdot 10^{-3}$	$8.3 \cdot 10^{-3}$	$8.3 \cdot 10^{-3}$
20	$5.3 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$	$5.5 \cdot 10^{-5}$	$1.9 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$
40	$7.7 \cdot 10^{-3}$	$3.8 \cdot 10^{-2}$	$3.5 \cdot 10^{-3}$	$1.1 \cdot 10^{-4}$	$4.6 \cdot 10^{-4}$	$4.9 \cdot 10^{-4}$

Table 3: Second technique results with n subdivisions in the x- and y-directions

$$\alpha_7^{**} = \alpha_6^{**}, \quad \alpha_8^{**} = \alpha_5^{**}.$$

We show our results obtained by using the above coefficients in Table 3.

Comparing Table 1 and Table 3, we see that the first error conversion technique (Table 1) is more accurate for relatively large ϵ values, but the second error conversion technique (Table 3) becomes as accurate as the first one for small ϵ.

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