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AN EXACT MACROSCOPIC EXTENDED MODEL WITH MANY MOMENTS

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Abstract: Extended thermodynamics (ET) is a very important theory: for example, it predicts hyperbolicity and finite speeds of propagation waves. Here its methods are applied to the balance equations suggested by the non relativistic limit of relativistic extended thermodynamics. This paper and the other one presented at this conference, that is "The Galilean relativity principle for a new kind of systems of balance equations in extended thermodynamics" are two parts of a same work and complete each other.

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1. Introduction

The non-relativistic limit of relativistic extended thermodynamics suggests to the balance equations (1) of paper [1] and shows also that the kinetic counterpart of the variables appearing in this system is

$$
\begin{aligned}
F^{i_1\cdots i_n} &= \int f c^{i_1}\cdots c^{i_n} d\underline{c} \quad , \quad F^{k i_1\cdots i_n} = \int f c^k c^{i_1}\cdots c^{i_n} d\underline{c}\,, \\
F_*^{i_1\cdots i_r} &= \int f c^{i_1}\cdots c^{i_r} (c^2)^{\frac{N+M+1-2r}{2}} d\underline{c}\,, \\
G^{k i_1\cdots i_r} &= \int f c^k c^{i_1}\cdots c^{i_r} (c^2)^{\frac{N+M+1-2r}{2}} d\underline{c}\,, \qquad (1)
\end{aligned}
$$

from which we see that:

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— All the tensors are symmetric,

— $F^{ki_1\cdots i_n}$ for n=0,..., N-1 is differentiated with respect to time in the subsequent equation,

— $F_*^{i_1\cdots i_M} = F^{ki_1\cdots i_N}\delta_{ki_{M+1}}\delta_{i_{M+2}i_{M+3}}\cdots\delta_{i_{N-1}i_N}$,

— $F_*^{i_1\cdots i_r} = G^{ki_1\cdots i_{r+1}}\delta_{ki_{r+1}}$ for $r=0,...,M-1$, (2)

— G^k is completely free.

The last 4 of the above conditions are called compatibility conditions.

From equations (1) we see also that if we consider two Galileanly equivalent frames we have the transformations for the various tensors which are reported in equation (2) of [1] with

$$X^{i_1\cdots i_n}_{j_1\cdots j_h} = \binom{n}{h}\delta^{(i_1}_{j_1}\cdots\delta^{i_h}_{j_h}v^{i_{h+1}}\cdots v^{i_n)}\,, \tag{3}$$

$$Y^{i_1\cdots i_r}_{j_1\cdots j_h} = \sum_{q_2=\sup\left\{h-\frac{N+M+1}{2},0\right\}}^{\inf\left\{\left[\frac{h}{2}\right],\frac{N+M+1-2r}{2}\right\}}\ \sum_{q_1=\sup\left\{0,h-\frac{N+M+1-2r}{2}-q_2\right\}}^{\inf\{r,h-2q_2\}}\binom{r}{q_1}$$
$$\frac{\left(\frac{N+M+1-2r}{2}\right)!}{q_2!(h-q_1-2q_2)!(q_1+q_2-h+\frac{N+M+1-2r}{2})!}$$
$$(v^2)^{q_1+q_2-h+\frac{N+M+1-2r}{2}}2^{h-q_1-2q_2}v_{(j_1}\cdots v_{j_{h-q_1-2q_2}}$$
$$\delta_{j_{h-q_1-2q_2+1}j_{h-q_1-2q_2+2}}\cdots\delta_{j_{h-q_1-1}j_{h-q_1}}\delta^{(i_1}_{j_{h-q_1+1}}\cdots\delta^{i_{q_1}}_{j_h)}v^{i_{q_1+1}}\cdots v^{i_r)}\,,$$

$$P^{i_1\cdots i_{r+1}}_{j_1\cdots j_h} = \sum_{q_2=\sup\left\{h-\frac{N+M+1}{2}-1,0\right\}}^{\inf\left\{\left[\frac{h}{2}\right],\frac{N+M+1-2r}{2}\right\}}\ \sum_{q_1=\sup\left\{0,h-\frac{N+M+1-2r}{2}-q_2\right\}}^{\inf\{r+1,h-2q_2\}}\binom{r+1}{q_1}$$
$$\frac{\left(\frac{N+M+1-2r}{2}\right)!}{q_2!(h-q_1-2q_2)!(q_1+q_2-h+\frac{N+M+1-2r}{2})!}$$
$$(v^2)^{q_1+q_2-h+\frac{N+M+1-2r}{2}}2^{h-q_1-2q_2}v_{(j_1}\cdots v_{j_{h-q_1-2q_2}}$$
$$\delta_{j_{h-q_1-2q_2+1}j_{h-q_1-2q_2+2}}\cdots\delta_{j_{h-q_1-1}j_{h-q_1}}\delta^{(i_1}_{j_{h-q_1+1}}\cdots\delta^{i_{q_1}}_{j_h)}v^{i_{q_1+1}}\cdots v^{i_{r+1})}\,,$$

$$Q^{i_1\cdots i_{r+1}}_{j_1\cdots j_{p+1}} = \sum_{q_2=\frac{N+M+1}{2}-p}^{\inf\left\{\left[\frac{N+M+2-p}{2}\right],\frac{N+M+1-2r}{2}\right\}}\ \sum_{q_1=\sup\left\{0,\frac{N+M+1}{2}-p-q_2+1+r\right\}}^{\inf\{r+1,N+M+2-p-2q_2\}}\binom{r+1}{q_1}$$

$$\frac{\left(\frac{N+M+1-2r}{2}\right)!}{q_2!(N+M+2-p-q_1-2q_2)!(q_1+q_2+p-\frac{N+M+3+2r}{2})!}$$
$$(v^2)^{q_1+q_2+p-\frac{N+M+3+2r}{2}} 2^{N+M+2-p-q_1-2q_2} v_{(j_1}\cdots v_{j_{N+M+2-p-q_1-2q_2}}$$
$$\delta_{j_{N+M+3-p-q_1-2q_2} j_{N+M+4-p-q_1-2q_2}} \cdots \delta_{j_{p-q_1} j_{p-q_1+1}} \delta^{(i_1}_{j_{p-q_1+2}} \cdots \delta^{i_{q_1}}_{j_{p+1})} v^{i_{q_1+1}} \cdots v^{i_{r+1})},$$

$$Z^{i_1\cdots i_r}_{j_1\cdots j_p} = \sum_{q_2=\frac{N+M+1}{2}-p}^{\inf\left\{\left[\frac{N+M+1-p}{2}\right],\frac{N+M+1-2r}{2}\right\}} \sum_{q_1=\sup\left\{0,\frac{N+M+1}{2}-p-q_2+r\right\}}^{\inf\{r,N+M+1-p-2q_2\}} \binom{r}{q_1}$$
$$\frac{\left(\frac{N+M+1-2r}{2}\right)!}{q_2!(N+M+1-p-q_1-2q_2)!(q_1+q_2+p-\frac{N+M+1+2r}{2})!}$$
$$(v^2)^{q_1+q_2+p-\frac{N+M+1+2r}{2}} 2^{N+M+1-p-q_1-2q_2} v_{(j_1}\cdots v_{j_{N+M+1-p-q_1-2q_2}}$$
$$\delta_{j_{N+M+2-p-q_1-2q_2} j_{N+M+3-p-q_1-2q_2}} \cdots \delta_{j_{p-q_1-1} j_{p-q_1}} \delta^{(i_1}_{j_{p-q_1+1}} \cdots \delta^{i_{q_1}}_{j_p)} v^{i_{q_1+1}} \cdots v^{i_r)}.$$

The proof of this property is here reported in Appendix.

2. The Entropy Principle and the Galilean Relativity Principle

These two principles are expressed in [1] by equations (5), by the transformations (2) and (8), by the definition (6), the consequence (7) and by the fact that h and ϕ'^k do not depend on v^j. By substituting equations $(2)_{1,2}$ into equation $(5)_1$ of [1], we obtain

$$dh = \sum_{n=0}^{N} \lambda_{i_1\cdots i_n} \sum_{h=0}^{n} X^{i_1\cdots i_n}_{j_1\cdots j_h} dF'^{j_1\cdots j_h} + dv^j \left\{ \sum_{n=0}^{N} \lambda_{i_1\cdots i_n} \sum_{h=0}^{n} \left(\frac{\partial}{\partial v_j} X^{i_1\cdots i_n}_{j_1\cdots j_h} \right) \right.$$
$$F'^{j_1\cdots j_h} + \sum_{r=0}^{M} \mu_{i_1\cdots i_r} \left[\sum_{h=0}^{N} \left(\frac{\partial}{\partial v_j} Y^{i_1\cdots i_r}_{j_1\cdots j_h} \right) F'^{j_1\cdots j_h} + \sum_{p=r}^{M} \left(\frac{\partial}{\partial v_j} Z^{i_1\cdots i_r}_{j_1\cdots j_p} \right) \right.$$
$$\left.\left. F'^{j_1\cdots j_p}_{*} \right] \right\} + \sum_{r=0}^{M} \mu_{i_1\cdots i_r} \left\{ \sum_{h=0}^{N} Y^{i_1\cdots i_r}_{j_1\cdots j_h} dF'^{j_1\cdots j_h} + \sum_{p=r}^{M} Z^{i_1\cdots i_r}_{j_1\cdots j_p} dF'^{j_1\cdots j_p}_{*} \right\}. \tag{4}$$

The Galilean relativity principle imply that the coefficient of dv^j must be equal to zero; so, by exchanging the order of summations, it remains

$$dh = \sum_{h=0}^{N} \left[\sum_{n=h}^{N} \lambda_{i_1\cdots i_n} X^{i_1\cdots i_n}_{j_1\cdots j_h} + \sum_{r=0}^{M} \mu_{i_1\cdots i_r} Y^{i_1\cdots i_r}_{j_1\cdots j_h} \right] dF'^{j_1\cdots j_h}$$

$$+\sum_{p=0}^{M}\sum_{r=0}^{p}\mu_{i_1\cdots i_r}Z^{i_1\cdots i_r}_{j_1\cdots j_p}dF'^{j_1\cdots j_p}_{*},$$

from which it follows equation $(9)_1$ of [1] and the decomposition (11) of [1] for the Lagrange multipliers. Moreover, the coefficient of dv^j in (4) must be zero and this implies equation $(10)_1$ of [1] after having exchanged the order of summations and used equations (4) and (11) of [1].

Similarly, by substituting equations $(3)_{1,2}$ in $(5)_2$ of [1] and by using also equation $(8)_2$ of [1], equation $(5)_2$ becomes

$$\begin{aligned}
&d\phi'^k + \underline{v^k dh} + h dv^k \\
&\quad = \sum_{n=0}^{N}\lambda_{i_1\cdots i_n}\left[\underline{v^k dF^{i_1\cdots i_n}} + F^{i_1\cdots i_n}dv^k + \sum_{h=0}^{n}X^{i_1\cdots i_n}_{j_1\cdots j_h}dF'^{kj_1\cdots j_h}\right] \\
&\quad + \sum_{r=0}^{M}\mu_{i_1\cdots i_r}\left[\underline{v^k dF^{i_1\cdots i_r}_{*}} + F^{i_1\cdots i_r}_{*}dv^k + \sum_{h=0}^{N}Y^{i_1\cdots i_r}_{j_1\cdots j_h}dF'^{kj_1\cdots j_h}\right. \\
&\left. + \sum_{p=r}^{M}Z^{i_1\cdots i_r}_{j_1\cdots j_p}dG'^{kj_1\cdots j_p}\right] + dv^j\left\{\sum_{n=0}^{N}\lambda_{i_1\cdots i_n}\sum_{h=0}^{n}\frac{\partial X^{i_1\cdots i_n}_{j_1\cdots j_h}}{\partial v_j}F'^{kj_1\cdots j_h} + \sum_{r=0}^{M}\mu_{i_1\cdots i_r}\right. \\
&\qquad\left. \times\left[\sum_{h=0}^{N}\frac{\partial Y^{i_1\cdots i_r}_{j_1\cdots j_h}}{\partial v_j}F'^{kj_1\cdots j_h} + \sum_{p=r}^{M}\frac{\partial Z^{i_1\cdots i_r}_{j_1\cdots j_p}}{\partial v_j}G'^{kj_1\cdots j_p}\right]\right\}.
\end{aligned}$$

By using equation $(5)_1$ of [1] the underlined terms can be eliminated, so that we have equation $(9)_2$ of [1]. Moreover, the coefficient of dv_j must be equal to zero, that is equation $(10)_2$ of [1] after having used equations (2) and (11) of [1].

The conditions which we have so found have been fully exploited in reference [2].

References

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Appendix

In order to find the tensors (3) let us firstly consider the tensor

$$\begin{aligned}
H^{i_1\cdots i_r} &= \int f c^{i_1}\cdots c^{i_r}(c^2)^\alpha d\underline{c} \qquad (5)\\
&= \int f(c'^{i_1}+v^{i_1})\cdots(c'^{i_r}+v^{i_r})(c'^2+2c'^i v_i+v^2)^\alpha d\underline{c'}\\
&= \sum_{q_1=0}^{r}\sum_{q_2+q_3\le\alpha}\binom{r}{q_1}\int f c'^{(i_1}\cdots c'^{i_{q_1}}v^{i_{q_1+1}}\cdots v^{i_r)}\frac{\alpha!}{q_2!q_3!\,(\alpha-q_2-q_3)!}\\
&(c'^2)^{q_2}2^{q_3}c'^{j_1}\cdots c'^{j_{q_3}}v_{j_1}\cdots v_{j_{q_3}}(v^2)^{\alpha-q_2-q_3}d\underline{c'}\\
&= \sum_{(q_1,q_2,q_3)\in\Delta}\binom{r}{q_1}\frac{\alpha!}{q_2!q_3!\,(\alpha-q_2-q_3)!}2^{q_3}\\
&\times(v^2)^{\alpha-q_2-q_3}H'^{e_1e_1\cdots e_{q_2}e_{q_2}j_1\cdots j_{q_3}(i_1\cdots i_{q_1}}v^{i_{q_1+1}}\cdots v^{i_r)}v_{j_1}\cdots v_{j_{q_3}}\,,
\end{aligned}$$

where we have used the relation $c^i = c'^i + v^i$ between Galileanly equivalent frames, also the binomial and trinomial rules for powers; moreover, $\sum_{(q_1,q_2,q_3)\in\Delta}$ means that the summation have to be done with respect to every tern of indexes (q_1, q_2, q_3) belonging to the set

$$\Delta = \left\{(q_1,q_2,q_3)\,:\,0\le q_1\le r\,,\,0\le q_2\,,\,0\le q_3\,,\,q_2+q_3\le\ \alpha\right\}.$$

With the following change of index from q_3 to h, defined by $q_3 = h - q_1 - 2q_2$, the set Δ converts into

$$\Delta' = \left\{(q_1,q_2,h)\,:\,0\le q_1\le r\,,\,0\le q_2\,,\,q_1+2q_2\le h\,,\,h-q_1-q_2\le\alpha\right\}.$$

Let us now transform suitably Δ'. The 1-st, 3-rd and 4-th inequalities defining it are:

$$0\le q_1\le r,\qquad q_1\le h-2q_2,\qquad h-q_2-\alpha\le q_1. \qquad (6)$$

The compatibilities between 1-st and 2-nd, 1-st and 3-rd, 2-nd and 3-rd of these are

$$0\le h-2q_2,\quad h-q_2-\alpha\le r,\quad h-q_2-\alpha\le h-2q_2,$$

or, by adding also the remaining 2-nd inequality which defines Δ',

$$q_2\le\left[\frac{h}{2}\right],\qquad h-\alpha-r\le q_2,\qquad q_2\le\alpha,\qquad 0\le q_2. \qquad (7)$$

After that, equations (6) become

$$\sup\{0, h-q_2-\alpha\} \leq q_1 \leq \inf\{r, h-2q_2\}.$$

The compatibility conditions between equations (7) are

$$h-\alpha-r \leq \left[\frac{h}{2}\right], \qquad 0 \leq h, \qquad h-\alpha-r \leq \alpha.$$

The first of these is a consequence of the others; in fact, from the 3-rd one we have $h \leq 2\alpha + r \leq 2\alpha + 2r$ from which the 1-st one follows when h is even. In the other case, h odd, we have still $h \leq 2\alpha + 2r$, as above, but it implies $h \leq 2\alpha + 2r - 1$ because h is odd; therefore the 1-st follows also in this case. Therefore, of the above relations it remains $0 \leq h \leq 2\alpha + r$.
After that equations (7) become

$$\sup\{0, h-\alpha-r\} \leq q_2 \leq \inf\left\{\left[\frac{h}{2}\right], \alpha\right\}.$$

Consequently, $\displaystyle\sum_{(q_1,q_2,q_3)\in\Delta}$ becomes $\displaystyle\sum_{h=0}^{2\alpha+r} \sum_{q_2=\sup\{0,h-\alpha-r\}}^{\inf\{[\frac{h}{2}],\alpha\}} \sum_{q_1=\sup\{0,h-q_2-\alpha\}}^{\inf\{r,h-2q_2\}}$.

Let now β and M be arbitrary integers such that $0 \leq \beta \leq 2\alpha + r$. We can split $\sum_{h=0}^{2\alpha+r}$ in $\sum_{h=0}^{\beta}$ and in $\sum_{h=\beta+1}^{2\alpha+r}$. In the first of these we use the change of index (from h to p) defined by $h = \beta + M + 1 - p$ and it becomes

$$\sum_{p=\beta+M+1-2\alpha-r}^{M} \sum_{q_2=\sup\{0,\beta+M+1-p-\alpha-r\}}^{\inf\{[\frac{\beta+M+1-p}{2}],\alpha\}} \sum_{q_1=\sup\{0,\beta+M+1-p-\alpha-q_2\}}^{\inf\{r,\beta+M+1-p-2q_2\}}$$

Consequently, equation (5) becomes

$$\begin{aligned} H^{i_1\cdots i_r} = & \sum_{h=0}^{\beta} \sum_{q_2=\sup\{0,h-\alpha-r\}}^{\inf\{[\frac{h}{2}],\alpha\}} \sum_{q_1=\sup\{0,h-q_2-\alpha\}}^{\inf\{r,h-2q_2\}} \binom{r}{q_1} \\ & \frac{\alpha!}{q_2!(h-q_1-2q_2)!\,(\alpha+q_2-h+q_1)!} 2^{h-q_1-2q_2} (v^2)^{\alpha+q_2-h+q_1} \\ & H'^{e_1e_1\cdots e_{q_2}e_{q_2}j_1\cdots j_{h-q_1-2q_2}(i_1\cdots i_{q_1}} v^{i_{q_1+1}} \cdots v^{i_r)} v_{j_1} \cdots v_{j_{h-q_1-2q_2}} \\ & + \sum_{p=\beta+M+1-2\alpha-r}^{M} \sum_{q_2=\sup\{0,\beta+M+1-p-\alpha-r\}}^{\inf\{[\frac{\beta+M+1-p}{2}],\alpha\}} \sum_{q_1=\sup\{0,\beta+M+1-p-\alpha-q_2\}}^{\inf\{r,\beta+M+1-p-2q_2\}} \binom{r}{q_1} \\ & \frac{\alpha!}{q_2!(\beta+M+1-p-q_1-2q_2)!\,(\alpha+q_2+q_1-\beta-M-1+p)!} \\ & 2^{\beta+M+1-p-q_1-2q_2} H'^{e_1e_1\cdots e_{q_2}e_{q_2}j_1\cdots j_{\beta+M+1-p-q_1-2q_2}(i_1\cdots i_{q_1}} v^{i_{q_1+1}} \cdots v^{i_r)} \end{aligned} \tag{8}$$

$$v_{j_1} \cdots v_{j_{\beta+M+1-p-q_1-2q_2}} (v^2)^{\alpha+q_2+q_1+p-\beta-M-1} .$$

Now we can draw the desired results of this preliminary work. From the definition (5) of $H^{i_1 \cdots i_r}$ and from (8),

— For $\alpha = 0$, $r = n \in [0, N]$, $\beta = n$, we obtain $(2)_1$ of [1] with $(3)_1$.

— For $\alpha = 0$, $r = N+1$, $\beta = N+1$, we obtain $(2)_1$ of [1] for $n = N+1$, with $(3)_1$.

— For $\alpha = \frac{N+M+1-2r}{2}$, $r \in [0, M]$, $\beta = N$, we obtain $(2)_2$ of [1] with $(3)_{2,5}$.

— Let us firstly write (5) and (8) with $r+1$ instead of r; after that, substitute $\alpha = \frac{N+M+1-2r}{2}$, $r \in [0, M]$, $\beta = N+1$, so obtaining $(2)_3$ of [1] with $(3)_{3,4}$.

This results are very interesting: if we call I the variables occurring in equation (1) of [1], and I' their counterparts in the other frame, we have found that I are expressed in terms of I' and no other moment has slipped in their relation! This fact confirms that our equations are the physically correct ones.