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MULTIPLE STATE OPTIMAL DESIGN PROBLEMS AND HASHIN-SHTRIKMAN BOUNDS

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Abstract: The Hashin-Shtrikman energy bounds are known to be saturated by sequential laminates. In this work we precisely determine the order of these laminates for the case of given sum of m energies, with m strictly less than the space dimension. Similar techniques for studying necessary conditions of optimality for multiple state optimal design problems are used.

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1. Introduction

In multiple state optimal design problems, one is trying to find the best arrangement of given materials, such that the obtained body has some optimal properties regarding m different regimes. To be more precise, in the most simple case of two isotropic materials, with conductivities α and β, one is trying to find a characteristic function χ on open and bounded $\Omega \subseteq \mathbf{R}^d$, such that the functional

$$J(\chi) = \int_\Omega F(\mathbf{x}, \mathbf{A}(\mathbf{x}), \mathsf{u}(\mathbf{x}))\, d\mathbf{x}$$

is minimal, where $\mathsf{u} = (u_1, \dots, u_m)$ is the state function determined by

$$\begin{cases} -\mathsf{div}\,(\mathbf{A}\nabla u_i) = f_i \\ u_i \in \mathrm{H}^1_0(\Omega)\,. \end{cases} \qquad i = 1, \dots, m$$

with $\mathbf{A} = \chi\alpha\mathbf{I} + (1-\chi)\beta\mathbf{I}$. Functions $f_i \in \mathrm{H}^{-1}(\Omega)$, $i = 1, \dots, m$, are given. The

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case $m = 1$ is studied by Murat and Tartar [5] and Tartar [7, 8]. Antonić and Vrdoljak [2, 3, 4] studied optimality conditions for the general case and showed interesting theoretical result simplifying numerical algorithm for the solution. We proved that an optimal design could be found among sequential laminates with matrix material α of order not greater than m.

Using a similar technique, in this article we study the question of optimality for Hashin-Shtrikman bounds. We consider the problem of optimising the sum of m energies:

$$\Phi(\mathbf{A}) = \sum_{i=1}^{m} \mathbf{A}\xi_i \cdot \xi_i \,, \tag{1}$$

for given vectors $\xi_1, \dots \xi_m \in \mathbf{R}^d$. The optimisation is taken over the set $\mathcal{K}(\theta)$ of all possible effective conductivities: matrices obtained by mixing isotropic phases α and β with local fraction θ of the first phase, mathematically described by the homogenisation theory introduced by Tartar [6, 8]. The set $\mathcal{K}(\theta)$ consists of all matrices with eigenvalues $\lambda_1, \dots, \lambda_d$ satisfying inequalities

$$\lambda_\theta^- \leq \quad \lambda_j \quad \leq \lambda_\theta^+ \quad (j = 1, \dots, d)\,, \tag{2}$$

$$\sum_{j=1}^{d} \frac{1}{\lambda_j - \alpha} \quad \leq \quad \frac{1}{\lambda_\theta^- - \alpha} + \frac{d-1}{\lambda_\theta^+ - \alpha}\,, \tag{3}$$

$$\sum_{j=1}^{d} \frac{1}{\beta - \lambda_j} \quad \leq \quad \frac{1}{\beta - \lambda_\theta^-} + \frac{d-1}{\beta - \lambda_\theta^+}\,, \tag{4}$$

where $\lambda_\theta^- = \left(\frac{\theta}{\alpha} + \frac{1-\theta}{\beta}\right)^{-1}$ and $\lambda_\theta^+ = \theta\alpha + (1-\theta)\beta$. We simplify the expression for Φ as done by Allaire in [1] (dot represents Euclid scalar product for matrices):

$$\Phi(\mathbf{A}) = \mathbf{A} \cdot \sum_{i=1}^{m} \xi_i \xi_i^\tau = \mathbf{A} \cdot \mathbf{\Xi}\,,$$

where $\mathbf{\Xi} = \sum_{i=1}^{m} \xi_i \otimes \xi_i$ is symmetric positive semi-definite matrix ($\otimes$ denotes the tensor product). The classical von Neumann result says that the optimal $\mathbf{A}$ is simultaneously diagonalisable with $\mathbf{\Xi}$, so our problem becomes optimisation of function

$$\phi(\lambda) = \sum_{i=1}^{d} \lambda_i \mu_i$$

over the set $\mathbf{\Lambda}(\theta) \subseteq \mathbf{R}^d$ given by (2)-(4). The spectrum $\{\mu_1, \dots, \mu_d\}$ of $\mathbf{\Xi}$ consists of at most $\min\{m, d\}$ positive eigenvalues (the others are equal to zero). Allaire [1] proved that, in a more general case of mixing two well ordered anisotropic materials, the optimal values for Φ are obtained by rank-d sequential laminates. Moreover, for the isotropic, two-dimensional case, explicit formula for calculating the extrema of function ϕ (Lemma 3.2.17) is given.

In this article, we concentrate on the case $m < d$, since it shows some interesting behaviour (Theorem 1). Moreover, for the case $m = 3$ (sum of three energies in general d-dimensional case), we have an explicit formula for this minimizing composite (Theorem 2).

Remark 1. Matrix $\mathbf{\Xi}$ is equivalently written as $\xi\xi^\tau$, where the matrix ξ consists of columns $\xi_1, \ldots, \xi_m$. Since $\mathrm{Im}(\xi) = \mathrm{Im}(\xi\xi^\tau)$ (for any matrix ξ), we have that $\widetilde{m} := \dim(\mathrm{span}\{\xi_1, \ldots, \xi_m\})$ equals the number of positive eigenvalues of $\mathbf{\Xi}$.

2. Main Result

Theorem 1. *For linearly independent $\xi_1, \ldots, \xi_m \in \mathbf{R}^d$, $m < d$, let $\Phi : \mathcal{K}(\theta) \to \mathbf{R}$ denote the sum of m energies:*

$$\Phi(\mathbf{A}) = \sum_{i=1}^{m} \mathbf{A}\xi_i \cdot \xi_i \,.$$

The maximum of function Φ (upper Hashin-Shtrikman bound) is obtained by a simple laminate, and the minimum (lower Hashin-Shtrikman bound) by a rank-m sequential laminate with matrix material $\alpha\mathbf{I}$.

For the idea of the proof, one should write down the Kuhn-Tucker system for optimisation of the function ϕ from Introduction, and use the fact that the minimum point belongs to the boundary of the set $\mathbf{\Lambda}(\theta)$. Analysing the system we conclude that at least $d-m$ eigenvalues λ_i should be equal to λ_θ^+. Moreover, we can easily eliminate parts of the boundary that do not satisfy equality in (3). For the maximum point, the result is correct even in the case of mixing anisotropic materials, although the characterisation of the set $\mathcal{K}(\theta)$ in this case is still unknown.

As stated by Theorem 1, strange asymmetry in α and β appears: Upper and lower Hashin-Shtrikman bounds are achieved by rank-m sequential laminates with matrix material α. In other words, function ϕ attains the same maximum and minimum on $\mathbf{\Lambda}(\theta)$ as well as on the bigger set given only by inequalities (2)-(3).

Corollary 1. *If vectors $\xi_1, \dots, \xi_m \in \mathbf{R}^d$ are linearly dependent, such that*

$$\dim(\mathrm{span}\{\xi_1, \dots, \xi_m\} = \widetilde{m} < d\,,$$

then the lower Hashin-Shtrikman bound is obtained by a rank-$\widetilde{m}$ sequential laminate with matrix material $\alpha\mathbf{I}$, and the upper Hashin-Shtrikman bound by a simple laminate.

3. The Case $m = 3$

As seen in previous sections, for the case $m = 3$, we have to minimize (maximization is trivial)

$$\phi(\lambda) = \sum_{i=1}^{3} \lambda_i \mu_i\,,$$

where $\mu_1 \geq \mu_2 \geq \mu_3 \geq 0$ are eigenvalues of matrix $\mathbf{\Xi}$ (the others are equal to 0). In this section we allow the possibility of linear dependence of vectors ξ_i, so μ_3, or both μ_3 and μ_2 can be equal to 0. In order to calculate explicitly a minimizer for the function ϕ, we pass to spherical coordinates. Of course, for the sake of minimization, the radial component of vector (μ_1, μ_2, μ_3) is irrelevant. The ordering of components μ_i implies that for angles $\varphi \in [0, 2\pi\rangle$ and $\vartheta \in [0, \pi]$ we have

$$\varphi \in \left[0, \frac{\pi}{4}\right]\,,\ \vartheta \leq \frac{\pi}{2}\,,\ \sin\varphi \geq \operatorname{ctg}\vartheta\,.$$

We introduce $\varphi_0 = \operatorname{arctg}\left(\dfrac{\lambda_\theta^- - \alpha}{\lambda_\theta^+ - \alpha}\right)^2$.

Theorem 2. *For given angles φ, ϑ we have the following possibilities for minimum point of ϕ:*

I. $\varphi \leq \varphi_0$: *rank-one laminate $(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+)$ is optimal,*

II. $\varphi > \varphi_0$: *We define c by $\dfrac{1}{c} = \dfrac{1}{\lambda_\theta^- - \alpha} + \dfrac{1}{\lambda_\theta^+ - \alpha}$ and calculate*

$$\begin{aligned}
\lambda_1 &= c\,(1 + \sqrt{\operatorname{tg}\varphi}) + \alpha\,,\\
\lambda_2 &= c\,(1 + \sqrt{\operatorname{ctg}\varphi}) + \alpha\,,\\
\vartheta_0(\varphi) &= \operatorname{arctg}\left((\lambda_\theta^+ - \alpha)^2\sqrt{(\lambda_1 - \alpha)^{-4} + (\lambda_2 - \alpha)^{-4}}\right)\,,
\end{aligned}$$

1. *$\vartheta \geq \vartheta_0(\varphi)$: rank-two sequential laminate $(\lambda_1, \lambda_2, \lambda_\theta^+)$ with matrix material $\alpha\mathbf{I}$ is optimal,*

2. $\vartheta < \vartheta_0(\varphi)$: *rank-three sequential laminate* $(\lambda_1, \lambda_2, \lambda_3)$ *with matrix material* $\alpha\mathbf{I}$ *is optimal, where*

$$(\lambda_3 - \alpha)^2 = \frac{\operatorname{tg}\vartheta}{\sqrt{(\lambda_1 - \alpha)^{-4} + (\lambda_2 - \alpha)^{-4}}} \ .$$

For the proof one should first consider the case $\vartheta = \frac{\pi}{2}$ $(\mu_3 = 0)$. Since this leads to optimisation over two-dimensional $\mathbf{\Lambda}(\theta)$, one can easily get the optimal microstructure: rank-one laminate if $\varphi \leq \varphi_0$ and rank-two sequential laminate with matrix material α if $\varphi > \varphi_0$. Next step is to decrease ϑ (increase μ_3): for $\vartheta \geq \vartheta_0(\varphi)$, the optimal microstructure remains the same, and for $\vartheta < \vartheta_0(\varphi)$ rank-two sequential laminate with matrix material α is optimal.

We can eliminate angles φ and ϑ by using equalities

$$\operatorname{tg}\varphi = \frac{\mu_2}{\mu_1} \quad \text{and} \quad \operatorname{tg}^2\vartheta = \frac{\mu_1^2 + \mu_2^2}{\mu_3^2} \tag{5}$$

and write down the result of the previous theorem in a more elegant way.

Corollary 2. *A. If* $\frac{\mu_2}{\mu_1} \leq \left(\frac{\lambda_\theta^- - \alpha}{\lambda_\theta^+ - \alpha}\right)^2$ *then rank-one laminate* $(\lambda_\theta^-, \lambda_\theta^+, \lambda_\theta^+)$ *is optimal.*

B. If not, calculate c, λ_1, λ_2 *and* λ_3 *by formulae given in Theorem 2, but using (5) to eliminate* φ *and* ϑ*. We have two possibilities:*

1. *If* $\lambda_3 \geq \lambda_\theta^+$ *then rank-two laminate* $(\lambda_1, \lambda_2, \lambda_\theta^+)$ *is optimal.*
2. *Else, rank-three laminate* $(\lambda_1, \lambda_2, \lambda_3)$ *is optimal*

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