International Journal of Pure and Applied Mathematics
Volume 42 No. 3 2008, 451-457

THE GALILEAN RELATIVITY PRINCIPLE FOR A NEW KIND OF SYSTEMS OF BALANCE EQUATIONS IN EXTENDED THERMODYNAMICS

M.C. Carrisi[1], S. Pennisi[2 §]
[1,2]Dipartimento di Matematica ed Informatica
Università degli Studi di Cagliari
72, Via Ospedale, Cagliari, 09124, ITALY
[1]e-mail: cristina.carrisi@tiscali.it
[2]e-mail: spennisi@unica.it

Abstract: The classical limit of relativistic extended thermodynamics with many moments suggests to consider a new kind of balance equations: The evolution equations for moments of order $n = 0, ..., N$ and suitable traces of some of higher order. In this paper we show the restrictions imposed to this system by the Galilean relativity principle, aided also by the entropy principle. To this end, we extend to our system a new method, found by S. Pennisi and T. Ruggeri for a less general case.

AMS Subject Classification: 26A33
Key Words: extended thermodynamics

1. Introduction

In paper [1] it has been shown that the non-relativistic limit of relativistic extended thermodynamics suggests to consider the balance equations

$$\begin{cases} \partial_t F^{i_1\cdots i_n} + \partial_k F^{ki_1\cdots i_n} = P^{i_1\cdots i_n} & \text{for } n = 0, ..., N, \\ \partial_t F_*^{i_1\cdots i_r} + \partial_k G^{ki_1\cdots i_r} = Q^{i_1\cdots i_r} & \text{for } r = 0, ..., M, \end{cases} \tag{1}$$

where N and M are two integers such that $N > M$ and $N + M$ is an odd number. Let us impose the Galilean relativity principle for this system. In [2] it is proved that, between two Galileanly equivalent frames we have the following transformations for the various tensors

Received: August 17, 2007
§Correspondence author

$$F^{i_1\cdots i_n} = \sum_{h=0}^{n} X^{i_1\cdots i_n}_{j_1\cdots j_h}(\underline{v})\, F'^{j_1\cdots j_h}\,, \tag{2}$$

$$F_*^{i_1\cdots i_r} = \sum_{h=0}^{N} Y^{i_1\cdots i_r}_{j_1\cdots j_h}(\underline{v})\, F'^{j_1\cdots j_h} + \sum_{p=r}^{M} Z^{i_1\cdots i_r}_{j_1\cdots j_p}(\underline{v})\, F_*'^{j_1\cdots j_p}\,,$$

$$G^{i_1\cdots i_{r+1}} = \sum_{h=0}^{N+1} P^{i_1\cdots i_{r+1}}_{j_1\cdots j_h}(\underline{v})\, F'^{j_1\cdots j_h} + \sum_{p=r}^{M} Q^{i_1\cdots i_{r+1}}_{j_1\cdots j_{p+1}}(\underline{v})\, G'^{j_1\cdots j_{p+1}}\,.$$

In [2] the explicit expressions of $X^{i_1\cdots i_n}_{j_1\cdots j_h}$, $Y^{i_1\cdots i_r}_{j_1\cdots j_h}$, $Z^{i_1\cdots i_r}_{j_1\cdots j_p}$, $P^{i_1\cdots i_{r+1}}_{j_1\cdots j_h}$ and $Q^{i_1\cdots i_{r+1}}_{j_1\cdots j_{p+1}}$ are reported from which the following properties hold

$$F^{ki_1\cdots i_n} - v^k F^{i_1\cdots i_n} = \sum_{h=0}^{n} X^{i_1\cdots i_n}_{j_1\cdots j_h}(\underline{v}) F'^{kj_1\cdots j_h}\,,$$

$$G^{ki_1\cdots i_r} - v^k F_*^{i_1\cdots i_r} = \sum_{h=0}^{N} Y^{i_1\cdots i_r}_{j_1\cdots j_h}(\underline{v}) F'^{kj_1\cdots j_h} + \sum_{p=r}^{M} Z^{i_1\cdots i_r}_{j_1\cdots j_p}(\underline{v}) G'^{kj_1\cdots j_p}\,, \tag{3}$$

$$\frac{\partial}{\partial v^j} X^{i_1\cdots i_n}_{j_1\cdots j_h} = \begin{cases} 0 & \text{for } n=h, \\ (h+1) X^{i_1\cdots i_n}_{j_1\cdots j_h j} & \text{for } n = h+1, ..., N, \end{cases} \tag{4}$$

$$\frac{\partial}{\partial v^j} Y^{i_1\cdots i_r}_{j_1\cdots j_h} = \begin{cases} (h+1) Y^{i_1\cdots i_r}_{j_1\cdots j_h j}\,, & \text{for } h = 0, ..., N-1, \\ (N+1) Z^{i_1\cdots i_r}_{(j_1\cdots j_M} \delta_{j_{M+1} j_{M+2}} \cdots \delta_{j_{N-2} j_{N-1}} \delta_{j_N j)} & \text{for } \ h = N. \end{cases}$$

$$\frac{\partial}{\partial v^j} Z^{i_1\cdots i_r}_{j_1\cdots j_p} = \begin{cases} 0 & \text{for } r=p, \\ (p-1)\delta_{(j_1 j_2} Z^{i_1\cdots i_r}_{j_3\cdots j_p)j} + (N+M+3-2p) & Z^{i_1\cdots i_r}_{(j_1\cdots j_{p-1}} \delta_{j_p)j}, \\ & \text{for } r = 0, ..., p-1 \end{cases}$$

The entropy principle for our system (1), with usual passages [3], is equivalent to assume the existence of Lagrange multipliers $\lambda_{i_1\cdots i_n}$ and $\mu_{i_1\cdots i_r}$ such that

$$\begin{aligned} dh &= \lambda_{i_1\cdots i_n} dF^{i_1\cdots i_n} + \mu_{i_1\cdots i_r} dF_*^{i_1\cdots i_r}\,, \\ d\phi^k &= \lambda_{i_1\cdots i_n} dF^{ki_1\cdots i_n} + \mu_{i_1\cdots i_r} dG^{ki_1\cdots i_r} \end{aligned} \tag{5}$$

plus a residual inequality. So we are now ready for the next section.

2. The Galilean Relativity Principle

2.1. The First Method

Let us now impose the Galilean relativity principle to our system by using the classical methodology. We have to decompose the quantities into their convective and non-convective parts. In order to do it we define

$$v^i = \frac{F^i}{F} \tag{6}$$

and we use equation (2), where the quantity $\underline{v}$ is defined in (6) and is not more the relative velocity between two frames. Moreover $F^{'\cdots}$, $F_*^{'\cdots}$ and $G^{'\cdots}$ are all non-convective quantities.

From equations $(2)_1$ and (6) we have that $F^{'j_1} = 0$. (7)

The decomposition of h and ϕ^k is $\quad h = h' \quad \phi^k = \phi^{'k} + hv^k\,,$ (8)

where h' and $\phi^{'k}$ are non-convective quantities. In [2] it is shown that, from these decompositions and from equation (5), it follows

$$\begin{aligned} dh &= \lambda'_{j_1\cdots j_h} dF^{'j_1\cdots j_h} + \mu'_{j_1\cdots j_p} dF_*^{'j_1\cdots j_p}\,, \\ d\phi^{'k} &= \lambda'_{j_1\cdots j_h} dF^{'kj_1\cdots j_h} + \mu'_{j_1\cdots j_p} dG^{'kj_1\cdots j_p}, \end{aligned} \tag{9}$$

$$\sum_{h=0}^{N-1}(h+1)\lambda'_{j_1\cdots j_h j}F^{'j_1\cdots j_h} + (N+1)\mu'_{(j_1\cdots j_M}\delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_N j_{N+1})}F^{'j_1\cdots j_N}\delta_j^{j_{N+1}}$$
$$+\sum_{p=1}^{M}\left[(p-1)\mu'_{jj_1\cdots j_{p-2}}\delta_{j_{p-1}j_p} + (N+M+3-2p)\mu'_{j_1\cdots j_{p-1}}\delta_{j_p j}\right]F_*^{'j_1\cdots j_p} = 0 \tag{10}$$

$$\sum_{h=0}^{N-1}(h+1)\lambda'_{j_1\cdots j_h j}F^{'j_1\cdots j_h k} + (N+1)\mu'_{(j_1\cdots j_M}\delta_{j_{M+1}j_{M+2}}\cdots\delta_{j_N j_{N+1})}$$
$$F^{'j_1\cdots j_N k}\delta_j^{j_{N+1}} + \sum_{p=1}^{M}\left[(p-1)\mu'_{jj_1\cdots j_{p-2}}\delta_{j_{p-1}j_p} + (N+M+3-2p)\mu'_{j_1\cdots j_{p-1}}\delta_{j_p j}\right]$$
$$G^{'j_1\cdots j_p k} + \left[\sum_{n=0}^{N}\lambda'_{i_1\cdots i_n}F^{'i_1\cdots i_n} + \sum_{r=0}^{M}\mu'_{i_1\cdots i_r}F_*^{'i_1\cdots i_r} - h\right]\delta^{kj} = 0,$$

$$\text{with} \quad \lambda'_{j_1\cdots j_h} = \sum_{n=h}^{N}\lambda_{i_1\cdots i_n}X^{i_1\cdots i_n}_{j_1\cdots j_h} + \sum_{r=0}^{M}\mu_{i_1\cdots i_r}Y^{i_1\cdots i_r}_{j_1\cdots j_h},$$

$$\mu'_{j_1\cdots j_p} \quad = \quad \sum_{r=0}^{p} \mu_{i_1\cdots i_r} Z^{i_1\cdots i_r}_{j_1\cdots j_p}. \tag{11}$$

Equations (10) express the condition that h and ϕ'^k do not depend on the velocity v^j. From equation (9), in the independent variables $F'^{j_1\cdots j_h}$ and $F'^{j_1\cdots j_p}_*$, we find

$$\lambda'_{j_1\cdots j_h} = \frac{\partial h}{\partial F'^{j_1\cdots j_h}} \quad \text{for } h \neq 1, \qquad \mu'_{j_1\cdots j_p} = \frac{\partial h}{\partial F'^{j_1\cdots j_p}_*}$$

which imply that $\lambda'_{j_1\cdots j_h}$, for $h \neq 1$, and $\mu'_{j_1\cdots j_p}$ are non-convective quantities.

Equation $(11)_1$ for $h = 1$ defines λ'_{j_1}. But we see in equation $(10)_1$ that $\lambda'_j F'$ appears for $h = 0$ and can be obtained from this equation in terms of quantities which are already proved to be non-convective, so that also λ'_j is non-convective too.

Let us define now $\widetilde{h}$ and $\widetilde{\phi}^k$ from

$$h \quad = \quad -\widetilde{h} + \lambda'_{j_1\cdots j_h} F'^{j_1\cdots i_h} + \mu'_{j_1\cdots j_p} F'^{j_1\cdots j_p}_*,$$
$$\phi'^k \quad = \quad -\widetilde{\phi}^k + \lambda'_{j_1\cdots j_h} F'^{j_1\cdots i_h k} + \mu'_{j_1\cdots j_p} G'^{j_1\cdots j_p k}.$$

Equations (9), with v_j, λ', $\lambda'_{j_1 j_2}$, ..., $\lambda'_{j_1\cdots j_N}$, and $\mu'_{j_1\cdots j_p}$ as independent variables becomes

$$d\widetilde{h} \quad = \quad F'^{j_1\cdots i_h} d\lambda'_{j_1\cdots j_h} + F'^{j_1\cdots j_p}_* d\mu'_{j_1\cdots j_p}$$
$$d\widetilde{\phi}^k \quad = \quad F'^{j_1\cdots i_h k} d\lambda'_{j_1\cdots j_h} + G'^{j_1\cdots j_p k} d\mu'_{j_1\cdots j_p},$$

which, taking into account also equation (7), are equivalent to

$$F' = \frac{\partial \widetilde{h}}{\partial \lambda'}, \quad \frac{\partial \widetilde{\phi}^k}{\partial \lambda'} = F'^{jk} \frac{\partial \lambda'_j}{\partial \lambda'}, \tag{12}$$

$$F'^{j_1\cdots j_n} = \frac{\partial \widetilde{h}}{\partial \lambda'_{j_1\cdots j_n}}, \quad \frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_n}} = F'^{jk} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_n}} + F'^{j_1\cdots j_n k} \text{ for } n = 2, \cdots, N,$$

$$F'^{j_1\cdots j_p}_* = \frac{\partial \widetilde{h}}{\partial \mu'_{j_1\cdots j_p}}, \quad \frac{\partial \widetilde{\phi}^k}{\partial \mu'_{j_1\cdots j_p}} = F'^{jk} \frac{\partial \lambda'_j}{\partial \mu'_{j_1\cdots j_p}} + G'^{j_1\cdots j_p k} \quad \text{for } p = 0, ..., M.$$

By using the above equations, the conditions (2) of [2] and equations (10) become

$$\frac{\partial \widetilde{\phi}^k}{\partial \lambda'} = \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \lambda'},$$

$$\frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_h}} = \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_h}} + \frac{\partial \widetilde{h}}{\partial \lambda'_{i_1\cdots i_h k}} \quad \text{for } h = 2, ..., N-1,$$

$$\frac{\partial \widetilde{h}}{\partial \mu'_{j_1\cdots j_M}} = \left(\frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_N}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_N}} \right) \delta_{k j_{M+1}} \delta_{j_{M+2} j_{M+3}} \cdots \delta_{j_{N-1} j_N} \,,$$

$$\frac{\partial \widetilde{h}}{\partial \mu'_{j_1\cdots j_r}} = \left(\frac{\partial \widetilde{\phi}^k}{\partial \mu'_{j_1\cdots j_{r+1}}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \mu'_{j_1\cdots j_{r+1}}} \right) \delta_{k j_{r+1}} \quad \text{for } r = 0, ..., M-1 \,,$$

$$\frac{\partial \widetilde{\phi}^{[k}}{\partial \lambda'_{j_1]\cdots j_N}} = \frac{\partial \widetilde{h}}{\partial \lambda'_{j[k}} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1]\cdots j_N}} \,,$$

$$\frac{\partial \widetilde{\phi}^{[k}}{\partial \mu'_{j_1]\cdots j_p}} = \frac{\partial \widetilde{h}}{\partial \lambda'_{j[k}} \frac{\partial \lambda'_j}{\partial \mu'_{j_1]\cdots j_p}} \quad \text{for } p = 0, ..., M$$

$$\sum_{h=2}^{N-1} (h+1) \lambda'_{j_1\cdots j_h j} \frac{\partial \widetilde{h}}{\partial \lambda'_{j_1\cdots j_h}} + \lambda'_j \frac{\partial \widetilde{h}}{\partial \lambda'}$$

$$+ (N+1) \mu'_{(j_1\cdots j_M} \delta_{j_{M+1} j_{M+2}} \cdots \delta_{j_N j_{N+1})} \frac{\partial \widetilde{h}}{\partial \lambda'_{j_1\cdots j_N}} \delta_j^{j_{N+1}}$$

$$+ \sum_{p=1}^{M} \left[(p-1) \mu'_{j j_1\cdots j_{p-2}} \delta_{j_{p-1} j_p} + (N+M+3-2p) \mu'_{j_1\cdots j_{p-1}} \delta_{j_p j} \right] \frac{\partial \widetilde{h}}{\partial \mu'_{j_1\cdots j_p}} = , 0$$

$$\sum_{h=2}^{N-1} (h+1) \lambda'_{j_1\cdots j_h j} \left(\frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_h}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_h}} \right) + 2 \lambda'_{j_1 j} \frac{\partial \widetilde{h}}{\partial \lambda'_{j_1 k}}$$

$$+ (N+1) \mu'_{(j_1\cdots j_M} \delta_{j_{M+1} j_{M+2}} \cdots \delta_{j_N j_{N+1})} \left(\frac{\partial \widetilde{\phi}^k}{\partial \lambda'_{j_1\cdots j_N}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \lambda'_{j_1\cdots j_N}} \right) \delta_j^{j_{N+1}}$$

$$+ \sum_{p=1}^{M} \left[(p-1) \mu'_{j j_1\cdots j_{p-2}} \delta_{j_{p-1} j_p} + (N+M+3-2p) \mu'_{j_1\cdots j_{p-1}} \delta_{j_p j} \right]$$

$$\times \left(\frac{\partial \widetilde{\phi}^k}{\partial \mu'_{j_1\cdots j_p}} - \frac{\partial \widetilde{h}}{\partial \lambda'_{jk}} \frac{\partial \lambda'_j}{\partial \mu'_{j_1\cdots j_p}} \right) + \widetilde{h} \delta^{kj} = 0 \,. \tag{13}$$

So we have to find the functions $\widetilde{h}$, $\widetilde{\phi}^k$ and λ'_j depending on λ', $\lambda'_{j_1 j_2}$,..., $\lambda'_{j_1\cdots j_N}$, $\mu'_{j_1\cdots j_p}$ subject to the above restrictions. After that the costitutive functions are given by $(12)_4$ for $n = N$ and by $(12)_6$.

2.2. The Second Method

We want now to simplify equations (13), by extending to our balance equations a method already used in [4] for a less general case. To this end, let us consider the following mathematical problem: Find the functions H and H^k of the variables $\lambda'_{j_1\cdots j_h}$ with $h = 0, ..., N$ and $\mu_{j_1\cdots j_p}$ with $p = 0, ..., M$ subject to the following restrictions:

$$
\begin{aligned}
&\frac{\partial H^k}{\partial \lambda'_{j_1\cdots j_h}} = \frac{\partial H}{\partial \lambda'_{j_1\cdots j_h k}} \quad \text{for } h = 0, ..., N-1\,,\\
&\frac{\partial H}{\partial \mu'_{j_1\cdots j_M}} = \frac{\partial H^k}{\partial \lambda'_{j_1\cdots j_N}} \delta_{k j_{M+1}} \delta_{j_{M+2} j_{M+3}} \cdots \delta_{j_{N-1} j_N}\,,\\
&\frac{\partial H}{\partial \mu'_{j_1\cdots j_r}} = \frac{\partial H^k}{\partial \mu'_{j_1\cdots j_{r+1}}} \delta_{k j_{r+1}} \quad \text{for } r = 0, ..., M-1\,,\\
&\frac{\partial H^{[k}}{\partial \lambda'_{j_1]\cdots j_N}} = 0\,,\\
&\frac{\partial H^{[k}}{\partial \mu'_{j_1]\cdots j_p}} = 0
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
&\sum_{h=0}^{N-1} (h+1) \lambda'_{j_1\cdots j_h j} \frac{\partial H}{\partial \lambda'_{j_1\cdots j_h}}\\
&\qquad + (N+1) \mu'_{(j_1\cdots j_M} \delta_{j_{M+1} j_{M+2}} \cdots \delta_{j_N j_{N+1})} \frac{\partial H}{\partial \lambda'_{j_1\cdots j_N}} \delta_j^{j_{N+1}}\\
&+ \sum_{p=1}^{M} \left[(p-1) \mu'_{j j_1 \cdots j_{p-2}} \delta_{j_{p-1} j_p} + (N+M+3-2p) \mu'_{j_1\cdots j_{p-1}} \delta_{j_p j} \right] \frac{\partial H}{\partial \mu'_{j_1\cdots j_p}}\\
&= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{h=0}^{N-1} (h+1) \lambda'_{j_1\cdots j_h j} \frac{\partial H^k}{\partial \lambda'_{j_1\cdots j_h}}\\
&\qquad + (N+1) \mu'_{(j_1\cdots j_M} \delta_{j_{M+1} j_{M+2}} \cdots \delta_{j_N j_{N+1})} \frac{\partial H^k}{\partial \lambda'_{j_1\cdots j_N}} \delta_j^{j_{N+1}}\\
&+ \sum_{p=1}^{M} \left[(p-1) \mu'_{j j_1 \cdots j_{p-2}} \delta_{j_{p-1} j_p} + (N+M+3-2p) \mu'_{j_1\cdots j_{p-1}} \delta_{j_p j} \right] \frac{\partial H^k}{\partial \mu'_{j_1\cdots j_p}}\\
&+ H \delta^{kj} = 0
\end{aligned}
$$

After that we define $\lambda'_j = \lambda'_j(\lambda', \lambda_{j_1 j_2}, \cdots, \lambda_{j_1\cdots j_N}, \mu'_{j_1\cdots j_p})$ implicitly defined by $\frac{\partial H}{\partial \lambda'_j} = 0$.

If we call $\widetilde{h}$ and $\widetilde{\phi}^k$ the functions H and H^k calculated for such value of

λ'_j it is easy to prove that, as consequence, they satisfy equations (13) and, consequently, they are the same functions of the first method. Uniqueness of the solution can also be proved. In such a way we have proved the equivalence of the two methods also for this new kind of system.

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