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MODELING AND SIMULATION FOR WATERSLIDES IN CAD FLUME SECTIONS

Kuang-Hua Chang
Williams Companies Foundation
School of Aerospace and Mechanical Engineering
University of Oklahoma
865 Asp Avenue, Norman, OK 73019, USA
e-mail: khchang@ou.edu

Abstract: This paper presents a modeling and simulation method for analyzing position, velocity, and acceleration of riding objects on recreational waterslides represented in computer-aided design (CAD) environment. Mathematical representations of a number of common flume sections are first created in parametric surfaces. A set of coupled differential equations based on Lagrange's equation of motion that describe the motion of the riding object are derived, in which friction forces are included. These second order differential equations are then solved using *Mathematica*. Initial position and velocity are specified for the entire waterslide, which is composed of basic flume sections. A different set of differential equations are solved for each section. The position and velocity of the riding object at the entrance of the following section are obtained from those at the exit of the previous neighboring section. A real-world waterslide configuration is presented to demonstrate the feasibility of the modeling and simulation method. The major contribution of the paper is extending waterslide simulation to true CAD-based flume sections, and bringing friction forces into the formulations that make the simulations more realistic.

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1. Introduction

Safety is the top priority in constructing recreational waterslides in amusement theme parks. Discovering safety problems after the waterslide is built and installed is usually too late and too costly to correct the problems. The first integrated modeling, analysis, and design method for safety of recreational waterslides has been developed [3]. In [3], the geometric shape of the waterslide is represented mathematically using B-spline surfaces (see [4]), which led to individualized flume sections that are too expensive to manufacture. The objective of this research is to construct flume sections in a computer-aided design (CAD) environment by using geometric dimensions such as height and width, to alleviate difficultly encountered in manufacturing flume sections.

In this paper the riding object is assumed a particle with concentrated mass. Friction forces between the riding object and the flume surface have been incorporated. With these assumptions, the ordinary differential equations that govern the motion of the riding object are derived using Lagrange's equations of motion, see [2]. These equations are then solved numerically using *Mathematica*, see [5]. Initial position and velocity are specified for the entire waterslide. Note that unlike the B-spline representations, different sets of differential equations are solved for respective flume sections represented in CAD. In solving different equations, the position and velocity of the riding object at the entrance of the following flume section are obtained from those at the exit of the current section. This approach also alleviates the problem of continuity requirement at the junctions of flume sections.

2. Mathematical Representations of Basic Flume Sections

Basic flume sections, such as straight, elbow, and curved, are created to serve as the building blocks for composing waterslide configurations, as shown in Figure 1. In addition, guard sections (essentially, vertical walls) are added to reinforce safety requirements, especially for elbow sections. Geometry of all sections is expressed in parametric surface forms in terms of the parametric coordinates u and w, using CAD geometric dimensions. The overall waterslide configuration can be expressed mathematically as

$$\overline{\boldsymbol{X}}(u,w) = \sum_{i}^{N} \boldsymbol{X}^i(u^i, w^i)\,, \tag{1}$$

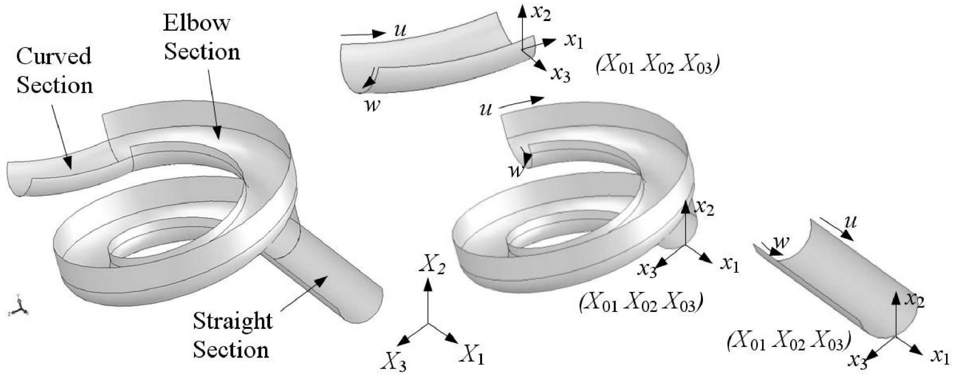

Figure 1: Geometric representation of waterslide in flume sections

where $\boldsymbol{X}^i(u^i,w^i)$ is the parametric equation of the i-th flume section, and N is the total number of sections. For each flume section (with superscript i removed for simplicity),

$$\boldsymbol{X}(u,w) = [X_1(u,w), X_2(u,w), X_3(u,w)]^T , \tag{2}$$

where $X_j(u,w)$ is the j-th coordinate of any given point on the surface with prescribed parameters (u,w). Note that these sections will be translated and properly oriented to compose an overall waterslide configuration by the following translation and rotation operations:

$$\boldsymbol{X}(u,w) = \boldsymbol{X}_0 + \boldsymbol{T}(\theta)\boldsymbol{x}(u,w) = \begin{bmatrix} X_{0_1} \\ X_{0_2} \\ X_{0_3} \end{bmatrix} + \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x_1(u,w) \\ x_2(u,w) \\ x_3(u,w) \end{bmatrix} \tag{3}$$

where $\boldsymbol{T}(\theta)$ is the rotational matrix that orients the section by rotating through angle θ about the X_2 axis, X_{0_i} is the location of the local coordinate system of the section in the waterslide configuration, $x(u,w)$ is the surface function of the section referring to its local coordinate system.

The mathematical representation of a flume section, for example, the elbow section shown in Figure 2, can be expressed using equation (3), where

$$\boldsymbol{x}(u,w) = [\cos u\phi(-R + r\cos w\pi), r(1-\sin w\pi) + H(1-u),$$
$$\sin u\phi(-R + r\cos w\pi + R]^T; \quad u,w \in [0,1] . \tag{4}$$

Note that in equation (4), R is the revolving radius, H is the height, and ϕ is the revolving angle measured along the x_2'-axis, which is offset along the x_3'-axis by amount R.

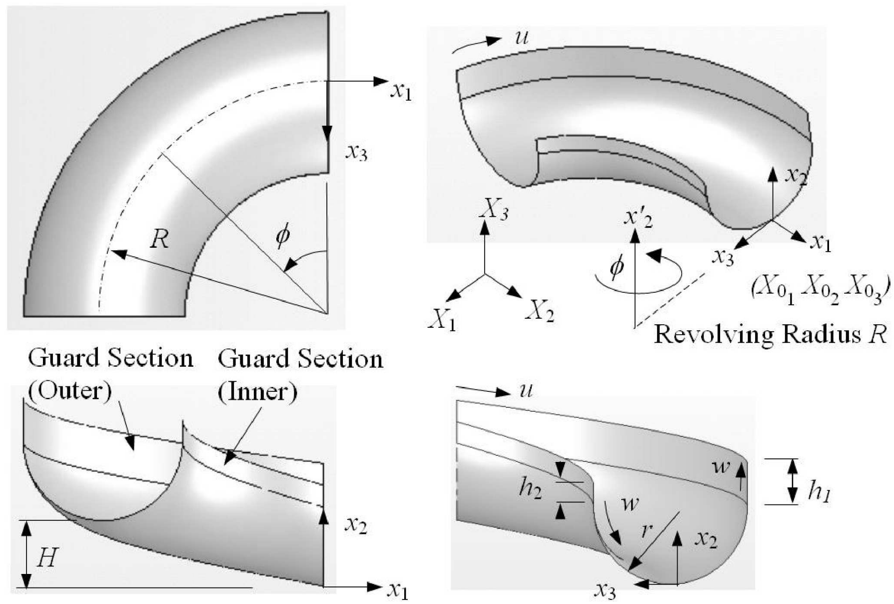

Figure 2: The Elbow flume section

In order to ensure the safety of the riding object, guard sections are often added to both sides of the elbow section, as shown in Figure 2. The parametric equations for the inner and outer guard sections are, respectively,

$$\boldsymbol{x}(u,w) = [\cos u\phi(-R-r) + wh_1 + H(1-u), \sin u\phi(-R-r) + R]^T; \text{ and} \tag{5a}$$

$$\boldsymbol{x}(u,w) = [\cos u\phi(-R-r) + wh_2 + H(1-u), \sin u\phi(-R-r) + R]^T; \tag{5b}$$

$u, w \in [0,1]$.

3. Equations of Motion

The well-known Lagrange's equation of motion based on Hamilton's Principle for particle dynamics (see [2]) can be stated as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = \boldsymbol{Q}, \tag{6}$$

where the Lagrangian function L is defined as $L \equiv T - V, \dot{\boldsymbol{q}} = \partial \boldsymbol{q}/\partial t$, and the generalized coordinates $\boldsymbol{q}$ in this waterslide application are the parametric coordinates of the surface; i.e., $\boldsymbol{q} = [u, w]^T$. When the system is conservative, $\boldsymbol{Q} = \boldsymbol{0}$. For a non-conservative system, $\boldsymbol{Q} = \boldsymbol{F}$, where $\boldsymbol{F}$ is the vector of generalized friction forces. For this motion analysis problem, the kinetic energy T and potential energy U are, respectively,

$$T = \frac{m}{2}\|\dot{\boldsymbol{X}}(u,w)\|^2, \text{ and } U = mgX_2(u,w)\,, \tag{7}$$

where m is the particle mass and g is the gravitational acceleration. For the friction cases, the generalized friction forces $\boldsymbol{Q} = [f_u, f_w]^T$can be derived as in [1]

$$f_u = -\mu(\boldsymbol{g} + \boldsymbol{a}_n)\cdot \boldsymbol{n}(\boldsymbol{e}_t \cdot \boldsymbol{X}_{,u}), \text{ and } f_w = -\mu(\boldsymbol{g} + \boldsymbol{a}_n)\cdot \boldsymbol{n}(\boldsymbol{e}_t \cdot \boldsymbol{X}_{,w})\,, \tag{8}$$

where μ is the friction coefficient; $\boldsymbol{n}$ is the unit normal surface vector; $\boldsymbol{a}_n$ is the normal acceleration of the riding object; and $\boldsymbol{e}_t$ is the unit vector along the tangential direction of the object's path, which is also the direction of the object's velocity $\dot{\boldsymbol{X}}$ and tangential acceleration $\boldsymbol{a}_t$.

Following equation (6), two coupled second order ordinary differential equations that govern the particle motion can be obtained as

$$k_0\ddot{u} = k_1\dot{u}^2 + k_2\dot{w}^2 + k_3\dot{u}\dot{w} + k_4\,, \tag{9a}$$

$$k_0\ddot{w} = k_5\dot{u}^2 + k_6\dot{w}^2 + k_7\ddot{u}\dot{w} + k_8\,, \tag{9b}$$

where k_0 to k_8 consist of polynomials of u and w and their products. Note that $\boldsymbol{X}$ must be at least second order differentiable with respect to u and w. These requirements are satisfied within individual flume sections, but not necessarily across the sections.

The initial conditions, including initial position and velocity of the riding object, must be provided in order to solve the equations of motion; i.e.,

$$u(0) = u^0, \quad w(0) = w^0, \quad \dot{u}(0) = \dot{u}^0, \quad \text{and} \quad \dot{w} = \dot{w}^0\,. \tag{10}$$

The system of ordinary differential equations can be solved numerically for positions $u(t)$ and $w(t)$, velocities $\dot{u}(t)$ and $\dot{w}(t)$, and accelerations $\ddot{u}(t)$ and $\ddot{w}(t)$, of the riding object using, for example, *Mathematica*. Since the individual section has its own mathematical representation, special numerical treatment must be applied to the area where two sections are joined. When the riding object moves closer to the end of the current section, the position and velocity of the object at that point are assumed as the initial position and velocity, respectively, of the following flume section. Mathematical representation of the following section will be employed to continue the computation. Note that the u-parameter value must be monitored closely in computation. When the u-

value is just over 1, the position and velocity are chosen as the initial position and velocity for the next section, in which the u-parameter value is set to 0. Certainly the exit u value is not always exactly 1. However, when the time step is refined, the error is reduced. The overall computation flow is illustrated in Figure 3.

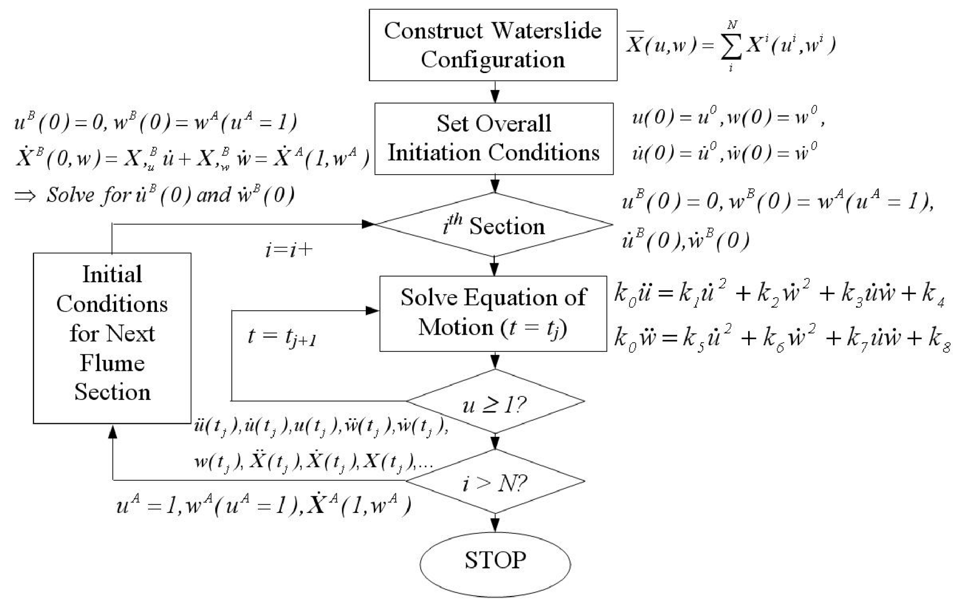

Figure 3: Computation flow for solving the equation of motion

4. Numerical Example

A waterslide configuration consisting of 20 flume sections shown in Figure 4 has been modeled and analyzed. The overall size of the waterslide is about 300 in. × 1,150 in. × 378 in. Note that the riding object starts at the center of the cross section ($w = 0.5$) of the top section. The friction coefficient is assumed $\mu = 0.08$.

The path of the riding object can be seen in Figure 4, in which the object runs over to the edge of the flume section at three critical areas A, B, and C. Note that when $w \leq 0$ and $w \geq 1$, the object is out of the section which poses a safety hazard to the rider. The design has to be revisited by either changing

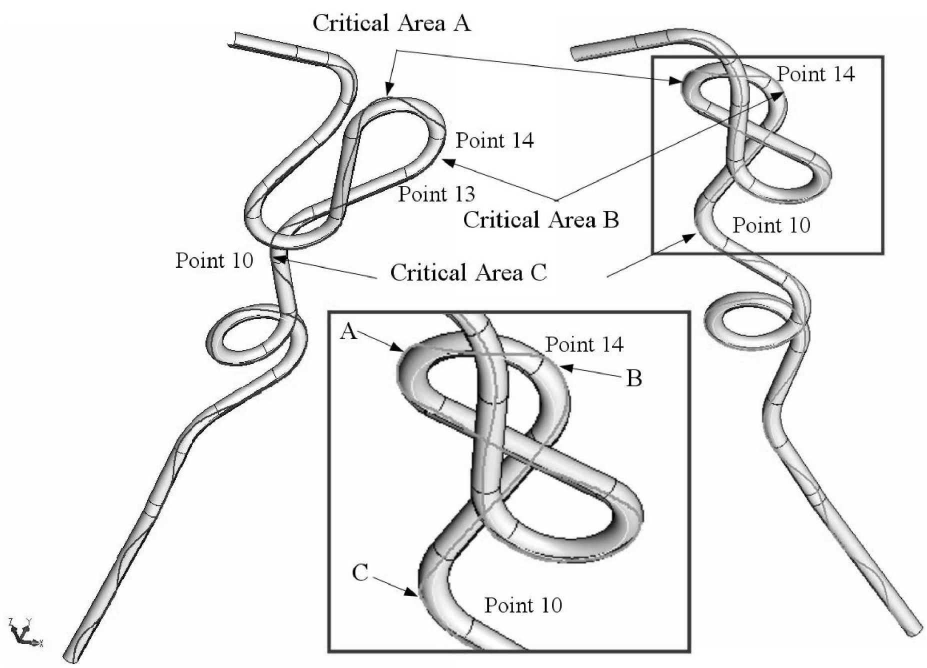

Figure 4: A waterslide example

the composition of the configuration or using closed (360 degrees) instead of open flume sections (180 degrees) currently employed. The overall riding time is 21.2 seconds which is very close to what was reported by the company who designed the waterslide (20 seconds). The maximum acceleration and velocity are 2.7 g and 12.8 MPH, respectively.

5. Conclusions and Future Research

In this paper, a computer-aided modeling and simulation method for analyzing motion of the riding objects on recreational waterslides has been presented. Mathematical representations of commonly employed sections have been created in parametric surfaces, which are compatible to and can be directly implemented into *CAD*. Equations of motion based on Lagrange's theory have been derived and solved using *Mathematica*. A different set of differential equations are solved for respective sections. A real-world waterslide configuration has been simulated. The results demonstrate the feasibility of the proposed modeling and simulation method. The simulation results can be used to verify design of the waterslides. In addition, design sensitivity analysis (DSA) that calculates the gradients of position, velocity, and acceleration of the riding object with respect to the design changes in *CAD* can be developed to support a systematic design approach.

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