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CLASSICAL AND QUANTUM ISING GAMES

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Abstract: We discuss a group of models relating to the Ising model, which we call games. A game represents an evolving design on a lattice or graph which is the temporal structure for the system, with a well-defined path corresponding to an actual experiment on the system. Time would be observed to vary in an ordinary incremental manner, associated with the path, while other lattice sites cannot be directly observed. We consider the evolution of the game in both a classical and quantum setting, and by coupling the game to a heat reservoir, consider the thermodynamics of the game.

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1. Introduction

In a previous paper [1], we introduced a simple statistical model, called the Sudoku model (and inspired by the puzzle by that name), intended to describe nonequilibrium processes. In this paper, we generalize these notions more directly to games. An ultimate goal of such a development would necessarily be to study simulations of games, and apply these ideas to variously crafted experimental systems.

The application of game theory to quantum mechanics started with [2] and [3]. A principal application of the ideas concerning quantum games has been to quantum computers, say as in [4].

We discuss the thermodynamics involved in a quantum game. There has been prior work done in this area, see [5].

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In the last two sections, we discuss quantum games, with the former section dedicated to an explicit discussion of such games and the latter to the thermodynamics. As it turns out, these aspects, quantum and thermodynamic, are complementary.

Finally, we make a few concluding remarks, and note that a more extensive paper is in preparation.

2. The Ising Model

The games we consider are somewhat patterned after the Ising model, a very simple equilibrium model for ferromagnetism, i.e. a classical system of spins on a lattice. We wish, in this section, merely to fix our notation for discussing the Ising model on a lattice (or more generally, a graph), since this is the notation we shall use in referring to games. The Ising spin system consists of a lattice L (assumed to be finite and of dimension d) of sites, each site (or vertex, in the case of a graph) being denoted by a letter, such as i, with an assignment of a spin value $s(i) = \pm 1$ at each site. The totality of spins, $C = \{s\}$, is a spin configuration for the lattice. We assume that there are N sites, and for discussions of critical phenomena [6], one frequently introduces an order parameter s, e.g.

$$s = \frac{1}{N}\sum_{i\in L} s(i) \ ,$$

where by $i \in L$ we simply mean that the sum is to be taken over all lattice sites. A typical Ising model has nearest-neighbor interactions, with a possible external field, and we represent this using a Hamiltonian:

$$H\{s\} = -J\sum_{i\in L}\sum_{j\in N(i)} s(i)\, s(j) + h\sum_{i\in L} s(i) \ ,$$

where $H\{s\}$ denotes the Hamiltonian for the configuration $\{s\}$, $J > 0$ (favoring aligned spins) is the exchange coupling and h is the external field. Here $N\{i\}$ denotes a special collection of sites associated with site i, which in the case of the nearest-neighbor Ising model consists merely of the sites j which are nearest-neighbors to i.

3. Games

If we take the long-time view, it is clear that the focus of much of game theory around the concept of equilibrium is well-justified. On the other hand, the aspects of games that we focus on are the designs and patterns that result in games, and the evolution of these patterns. In fact, we neglect entirely the rational aspect of games, and the idea that there are players involved in games.

The language we use is that of lattices or graphs, instead of players and decisions. Thus, at time t, the part of the game pattern being considered is called a vertex, $\mathbf{v}_t$. This vertex represents information, and ancillary information is recorded as a field ψ_t associated with the vertex $\mathbf{v}_t$. Let us take the game of chess to illustrate this. At time t, there are certain pieces that can be selected from, and each piece has certain possible moves or can make certain captures. The particular piece that is selected, and other information about the move, such as whether it was involved in a capture, is recorded in the field ψ_t, which we can simply formulate as a numerical vector, and the square that the piece was moved to is recorded as the vertex $\mathbf{v}_t$. Because there are six categories for the type of chess piece, one representation of the field would be a six-dimensional vector, with zeros in all positions but one, and the number in that position representing a label for the piece that was moved. We regard ψ_t as characterizing the state of the system at time t. We can regard time t as a discrete variable, with each successive time separated from the next by a time interval, Δt, which for simplicity might be regarded as a constant. The evolution of the game from beginning to end can be expressed as a time series $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$, which is actually a visual path on the chessboard that corresponds to a time sequence. We call this visual representation a time path. In addition to $\mathbf{v}_t$, we have an entire chessboard configuration at time t, yielding an additional structure to time (actually numerous other time series, related to the one we specify in the particular time path). In this view, time is three-dimensional, loosely speaking.

The interweaving and interlocking of the various time series signifies a mathematical design, and allows us to formalize a game as an evolution of designs, the designs being temporal designs inherent in time series. We can focus on the vertices and fields, with no direct reference to the games at all, and think of this in abstract terms as a type of mathematical model.

4. Hamiltonians

Let us ignore the external field, h, in the description of the Ising model, and note that since $|s(i)| = 1$, the statistical mechanics of the model is equivalent to that with Hamiltonian,

$$H = \frac{J}{2} \sum_i \sum_{\mathbf{u}} \left(s(i + \mathbf{u}) - s(i) \right)^2 ,$$

where the first sum is over all lattice sites i and the second symbolizes a sum over all $i + \mathbf{u}$, these sites being nearest-neighbors to i. This is obviously generalizable to games as

$$H = \frac{J}{2} \sum_i \sum_{\mathbf{u}} \left(\psi_{i+\mathbf{u}} - \psi_i \right)^2 ,$$

where we allow $i + \mathbf{u}$ to symbolize other sites besides just nearest-neighbors. Note that i denotes a time. We permit Hamiltonians to have long-range interactions (in time). A classical structure for the game can then be elaborated using the Hamiltonian to introduce randomness, see [1].

5. Quantum Games

We will now introduce quantum mechanics into games (see [1] for details on classical games). We can picture an underlying graph or lattice associated with the time path, with both edges and vertices (possibly many vertices not included in the time path). This allows us to identify at least two types of time, one associated with the real time path, that we can call t, and another τ, which is an independent time variable.

Conventional quantum mechanics supplies us with much of the relevant structure, so we will be rather terse. If we establish a sequence of measurements using two (noncommuting) observables $\mathbf{A}$ and $\mathbf{B}$ (i.e. Hermitian operators), we establish a random time evolution, just as one might have for a classical game evolving by rules that utilize randomness. It is straightforward to see that such quantum games can lead to frustration.

We use unitary operators $\mathbf{U}$ for time evolution when measurements are not being made, these operators assumed to have the form $\exp(iH_t \Delta t/\hbar)$, with Δt being interpreted as a small time interval between vertices, the current Hamiltonian H_t assumed to be some Hermitian operator, and $\hbar$ being, of course, Planck's constant (divided by 2π). We can combine time evolution and measurement as a sequence of time evolution operators with interspersed observables, acting on some initial state ψ.

6. Thermodynamics

We can work with temperature in quantum games, just as in classical games. Imagine, as in that situation, that the system is coupled to a heat reservoir, so that temperature is defined at each time step. We can then introduce thermodynamics. There is also another approach that is open to us when using quantum mechanics.

If $\mathbf{v}$ is a vertex not on the time path, there is no special or preferred path associated with $\mathbf{v}$. Therefore, to place this in context, in the quantum game, we must consider each directed path P that contains this vertex. We restrict our attention to paths that have initial vertices for which the state ψ is specified. The time evolution operators $\mathbf{U}$ on such a path, when we are dealing with some vertex off the time path, need not be unitary. By considering all possible paths, we obtain, through this evolution, an ensemble of state vectors for the vertex $\mathbf{v}$.

Considering time evolution from an initial vector let $\psi_{\mathbf{v}P}$ be the unique vector at $\mathbf{v}$ for the path P, and let $\mathbf{U}$ be an evolution operator (we are not considering measurables). For simplicity, let us assume that $\psi_{\mathbf{v}P}$ is an eigenvector of $\mathbf{U}$, and

$$\mathbf{U}\psi_{\mathbf{v}P} = \exp\left(i\omega\Delta t - \beta E\right)\psi_{\mathbf{v}P}$$

with ω and E certain real values, and $\beta = 1/k_B T$, T being the temperature. By considering all paths that feed into the current vertex (and include no vertices from the future), we will obtain a sum of products of exponential factors $\exp\left(i\omega\Delta t - \beta E\right)$. In this way we see that there may be a way of introducing thermodynamics into quantum games.

7. Concluding Remarks

We have been interested in presenting games as evolving designs, and only secondarily associated with such optimality functions as Hamiltonians. We have patterned the games after the Ising model, and to understand better why we refer to them as games, the discussion of the Sudoku model in [1] might be helpful. The intent is to use games as simple models for nonequilibrium processes. We discussed this both in a classical and in a quantum context. In both cases, the systems can evolve to a frustrated pattern. There is a great deal that cannot be clarified in a short paper like this, and a more extensive discussion is in preparation.

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