International Journal of Pure and Applied Mathematics
Volume 42 No. 4 2008, 475-480

MIMIMUM ENERGY CONTROL AND ITS APPLICATIONS

Akira Ichikawa
Department of Aeronautics and Astronautics
Kyoto University
Kyoto, 606-8501, JAPAN
e-mail: ichikawa@kuaero.kyoto-u.ac.jp

Abstract: For a stabilizable linear system, the minimum energy problem to steer the initial state to the origin asymptotically is considered, where the energy is understood in the $\mathcal{L}_2$ sense. The infimum is obtained in terms of the maximal solution of a singular Riccati equation, and suboptimal feedback controllers are designed. As an application, a circular restricted three-body (Earth-Moon-spacecraft) problem is considered, and the Halo orbit transfer near the L_2 Lagrangian point is discussed.

AMS Subject Classification: 93B51
Key Words: minimum energy control, periodic solutions, optimal regulator, singular Riccati equation

1. Minimum Energy Problem

Consider the linear system

$$\dot{x} = Ax + Bu, \tag{1}$$

where $x \in R^n$ and $u \in R^m$. Let $x(t; x_0, u)$ be the solution of (1) with initial condition $x(0) = x_0$. The Euclidean norm of vectors is denoted by $|\cdot|$ and the set of all eigenvalues of A by $\sigma(A)$.

For each x_0, let $u \in \mathcal{L}_2(0, \infty; R^m)$ be such that $x(t; x_0, u) \to 0$ as $t \to \infty$. Such a control is said to be admissible, and the set of all admissible controls for x_0 is denoted by $\mathcal{U}(x_0)$. If (A, B) is stabilizable, this set is nonempty. Define

$$E(x_0) = \inf\{\| u \|_2^2 : u \in \mathcal{U}(x_0)\},$$

where $\| \cdot \|_2$ denotes the norm in $\mathcal{L}_2$ space.

Received: August 17, 2007

To find $E(x_0)$ explicitly, assume first that (A, B) is controllable, and consider the algbraic Riccati equation

$$A'X + XA + Q - XBB'X = 0, \ \ Q \geq 0. \tag{2}$$

Let $\mathcal{U}(T; x_0)$ be the set of $u \in \mathcal{L}_2(0, T; R^m)$ such that $x(T; x_0, u) = 0$. The following theorem is known Priola et al [3].

Theorem 1.1. *Suppose (A, B) is controllable. Then for each $Q \geq 0$ there exists a maximal solution $X \geq 0$ of (2) such that*

$$x_0'Xx_0 = \inf_{T>0} \inf_{\mathcal{U}(T;x_0)} \int_0^T [x(t; x_0, u)'Qx(t; x_0, u) + |u(t)|^2]dt.$$

Let X_ϵ be the stabilizing solution of (2) with Q replaced by $Q + \epsilon I$, $\epsilon > 0$.

Lemma 1.1. *Suppose (A, B) is stabilizable. Then $X_\epsilon \geq X$ for any symmetric solution X of (2).*

Proof. The difference $X_\epsilon - X$ satisfies the equation

$$(A - BB'X_\epsilon)'(X_\epsilon - X) + (X_\epsilon - X)(A - BB'X_\epsilon) \\ + \epsilon I + (X_\epsilon - X)BB'(X_\epsilon - X) = 0.$$

Because $A - BB'X_\epsilon$ is stable, $X_\epsilon - X \geq 0$. □

Lemma 1.2. *X_ϵ is monotone decreasing as $\epsilon \to 0$, and its limit X is the maximal solution of (2).*

To characterize $E(x_0)$, consider the singular Riccati equation

$$A'X + XA - XBB'X = 0. \tag{3}$$

Theorem 1.2. *Suppose that (A, B) is stabilizable. Then*

$$E(x_0) = \inf_{\mathcal{U}(x_0)} \| u \|_2^2 = x_0'Xx_0,$$

where X is the maximal solution of (3).

Proof. Consider the cost function

$$J^\epsilon(u; x_0) = \int_0^\infty [\epsilon|x(t)|^2 + |u(t)|^2]dt.$$

Since $\| u \|_2^2 \leq J^\epsilon(u; x_0)$ for any $u \in \mathcal{U}(x_0)$, $E(x_0) = \inf \| u \|_2^2 \leq \inf J^\epsilon(u; x_0) = x_0'X_\epsilon x_0$, where X_ϵ is the stabilizing solution of (2) with $Q = \epsilon I$. Pass to the limit $\epsilon \to 0$ to obtain $E(x_0) \leq x_0'Xx_0$.

To prove the converse inequality, note that for the maximal solution of the

Riccati equation (3) the equality

$$\int_0^T |u(t)|^2 dt + x'(T)Xx(T) = x_0'Xx_0 + \int_0^T |u + B'Xx|^2 dt$$

holds for any $u \in \mathcal{L}_2(0, T; R^m)$. Choose $u \in \mathcal{U}_(x_0)$ to obtain $\| u \|_2^2 \geq x_0'Xx_0$, and hence $E(x_0) \geq x_0'Xx_0$. □

When the maximal solution of (3) is $X = 0$, a new definition is introduced by Priola et al [3].

Definition 1.1. The system (1) is said to be null controllable with vanishing energy (NCVE for short) if for each initial $x(0) = x_0$ there exists a sequence of pairs (T_N, u_N), $0 < T_N \uparrow \infty$, $u_N \in L_2(0, T_N; R^m)$ such that $x(T_N; x_0, u_N) = 0$ and $\| u_N \|_2 \to 0$.

Necessary and sufficient conditions for NCVE are given as follows Priola et al [3].

Theorem 1.3. *The following statements are equivalent:*

(i) (A, B) *is NCVE.*

(ii) (A, B) *is controllable, and* $X = 0$ *is the maximal solution of the algebraic Riccati equation* (3).

(iii) (A, B) *is controllable, and* $Re(\lambda) \leq 0$ *for any* $\lambda \in \sigma(A)$.

In view of Theorem 1.2, the notion of NCVE can be relaxed:

Definition 1.2. The system (1) is said to be stabilizable with vanishing energy (SVE) if $E(x_0) = 0$ for all x_0.

Theorem 1.4. (A, B) *is SVE if and only if:*

(a) (A, B) *is stabilizable, and*

(b) $Re(\lambda) \leq 0$ *for any* $\lambda \in \sigma(A)$.

Proof. It is enough to note that the controllable part of the system is NCVE.

Let A have pure imaginary eigenvalues $\pm j\omega$. Then (1) with $u = 0$ has periodic solutions. Each periodic solution determines an orbit. For a given orbit $\mathcal{O}$, the minimization problem $\min_{x_0 \in \mathcal{O}} E(x_0)$ and designing feedback controls whose $\mathcal{L}_2$-norms are close to $E(x_0)$ are important in space vehicle control. Since $E(x_0) = x_0'Xx_0$, $E(x_0)$ is a continuous function of x_0, and the minimum of $E(x_0)$ over any periodic orbit $\mathcal{O}$ (a compact set) exists. Its search is a constrained minimization. Now let x_0^* be the minimizing point on the orbit. Apply the feedback law $u = -B'X_\epsilon x$ with sufficiently small $\epsilon > 0$, when the state reaches x_0^*. Then it steers x_0^* asymptotically to the origin, and $E(x_0) \leq \| u \|_2^2 \leq x_0' X_\epsilon x_0$. Hence this is a suboptimal control. In the next section, the minimization of $E(x_0)$ for the Earth-Moon-spacecraft system is discussed in detail. □

The notion of NCVE is defined for discrete-time systems and periodic systems in Ichikawa [1], Ichikawa [2], and the relative orbit transfer along an elliptical orbit is considered in Shibata et al [4].

2. Halo Orbit Transfer near L_2 Point

Consider the Earth-Moon-spacecraft system regarded as the circular restricted three-body problem, where the Earth-Moon system, by assumption, rotates with a constant angular velocity ($\omega = 2.661699 \times 10^{-6}$ rad/s) about their composite center of mass, and their orbital motion is not affected by the spacecraft. Then the equations of motion are nonlinear, and possess five equilibrium points $L_1 - L_5$ known as Lagrangian points, see Wie [5].

The $L_2(-1.15568, 0, 0)$ point is called translunar point, and the linearized equations of motion about this point in nondimensional form are given by

$$\begin{aligned} \ddot{x} - 2\dot{y} - (2\sigma + 1)x &= u_x, \\ \ddot{y} + 2\dot{x} + (\sigma - 1)y &= u_y, \\ \ddot{z} + \sigma z &= u_z, \end{aligned} \tag{4}$$

where $\sigma = 3.19043$. The equations for x and y are independent of z and determine the in-plane motion, and its characteristic equation has two real and two imaginary roots: ± 2.15868 and $\pm 1.86265j$. Thus the L_2 point is unstable, but the system (4) has periodic solutions. The $L_1(-0.83692, 0, 0)$ and L_2 points are of practical importance for future space missions involving the stationing of a communication platform or a lunar space station. For lunar far-side communications, it is desirable to maintain a 3500-km halo orbit (periodic trajectory) about the L_2 point. Motivated by this, the orbit transfer of the in-plane motion will be discussed.

The state space equation is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2\sigma + 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1 - \sigma & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}, \tag{5}$$

where $\mathbf{x} = \begin{bmatrix} x & \dot{x} & y & \dot{y} \end{bmatrix}'$ and $\mathbf{u} = \begin{bmatrix} u_x & u_y \end{bmatrix}'$. The periodic solution of this equation is given by

$$x(t) = a\sin(\omega t + \alpha_0), \tag{6}$$

$$y(t) = \gamma a\cos(\omega t + \alpha_0), \tag{7}$$

where $\omega = 1.86265$, $a = (x_0^2 + \dot{x}_0^2/\omega^2)^{1/2}$, $\gamma = (\omega^2 + 2\sigma + 1)/2\omega = 2.91260$, and α_0 is determined by $\cos\alpha_0 = \dot{x}_0/a\omega$ and $\sin\alpha_0 = x_0/a$. In view of (6), (7), the periodic orbit $\mathcal{O}$ is an ellipse

$$(x/a)^2 + (y/\gamma a)^2 = 1.$$

The period of this solution is $\tau = 2\pi/\omega = 3.37154$, which corresponds to the actual period 14.6607 (days). The initial condition for (5) is given by $\mathbf{x}_0 = \begin{bmatrix} x_0 & \dot{x}_0 & \gamma/\omega\ \dot{x}_0 & -\gamma\omega\ x_0 \end{bmatrix}'$.

The maximal solution of the singular Riccati equation approximated by X_ϵ with $\epsilon = 10^{-14}$ is given by

$$\begin{bmatrix} 36.1240 & 10.56528 & -6.75656 & 6.65862 \\ 10.5652 & 3.08999 & -1.97609 & 1.94744 \\ -6.75656 & -1.97609 & 1.26373 & -1.24541 \\ 6.65862 & 1.94744 & -1.24541 & 1.22736 \end{bmatrix},$$

and its eigenvalues are $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (41.7051, 0, 0, 0)$ and the eigenvector corresponding to λ_1 is

$$p_1' = \begin{bmatrix} 0.930687 & 0.272197 & -0.174074 & 0.171550 \end{bmatrix}.$$

Now $E(\mathbf{x}_0) = \mathbf{x}_0' X \mathbf{x}_0 = q^2 p_1' X p_1 = \lambda_1 q^2$, where $q = \mathbf{x}_0' p_1$. Hence the minimization of q^2 subject to $x_0^2 + \dot{x}_0^2/\omega^2 = a^2$ yields the optimal initial position. The optimal initial condition is given by

$$x_0^* = \pm\alpha/(\alpha^2 + \beta^2\omega^2)^{1/2}\ a,$$

$$\dot{x}_0^* = \pm\beta\omega^2/(\alpha^2 + \beta^2\omega^2)^{1/2}\ a.$$

The minimum of q^2 is given by $q^{*2} = (\alpha^2 + \beta^2\omega^2)a^2$, and the minimum of $E(\mathbf{x}_0)$ by $E(\mathbf{x}_0^*) = \mathbf{x}_0^{*\prime} X \mathbf{x}_0{}^* = \lambda_1 q^{*2}$. Since $\alpha = 1.12251 \times 10^{-6}$ and $\beta = 1.09315 \times 10^{-6}$, the optimal initial values are $(x_0^*, \dot{x}_0^*) = (0.482784a, 1.60120a)$. Moreover, $q^{*2} = 5.40594 \times 10^{-12} a^2$, and $E(\mathbf{x}_0^*) = 2.25453 \times 10^{-10} a^2$.

Note that the transfer problem from a periodic orbit to another is reduced to the problem discussed above.

Acknowledgments

This work is partly supported by the Ministry of Education, Culture, Sports, Science and Technology, Japan, under Grant-in-Aid for Scientific Research (C), No. 19560785.

References

[1] A. Ichikawa, Null controllability with vanishing energy for discrete-time systems, *Systems and Control Letters*, To Appear.

[2] A. Ichikawa, Null controllability with vanishing energy for discrete-time systems in Hilbert space, *SIAM J. Control Optim.*, **46**, No. 2 (2007), 683-693.

[3] E. Priola, J. Zabczyk, Null controllability with vanishing energy, *SIAM J. Control Optim.*, **42**, No. 3 (2003), 1013-1032.

[4] M. Shibata, A. Ichikawa, Orbital rendezvous and flyaround based on null controllability with vanishing energy, *Journal of Guidance, Control, and Dynamics*, **30**, No. 4 (2007), 934-945.

[5] B. Wie, *Space Vehicle Dynamics and Control*, American Institute of Aeronautics and Astronautics (1998).