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ON WEAK DOMINATION IN GRAPHS

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Abstract: A subset $S \subseteq V(G)$ is a weak dominating set of a graph G if for any vertex $y \in V(G) - S$ there exists a vertex $x \in S$ adjacent to y and such that $\deg_G(x) \leq \deg_G(y)$. A minimal weak dominating set S of G is a weak dominating set that contains no weak dominating set of G as a proper subset. The total number of the mentioned dominating sets for some classes of graphs is determined.

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1. Introduction

In general we use the standard terminology and notation of graph theory, see [2]. Only simple undirected graphs are considered. A *tree* T is a connected graph with no cycles. P_n denotes a *path* on n vertices. $\overline{K_n}$ stands for the complement of K_n, i.e. the graph consisting of n isolated vertices. A *pendant vertex* is vertex of degree one. A vertex of degree one in a tree we call a *leaf*. By $K_{1,n-1}$ we denote the n-vertex *star*. Given integers n and m with $m \geq 3$ and $n \geq m+1$. The *palm* $P_{n,m}$ is the graph obtained from two vertex disjoint graphs $K_{1,m-1}$ and P_{n-m} by adding an edge joining the center of $K_{1,m-1}$ to an endvertex of P_{n-m}. Let G, H be graphs with $|V(G)| = n$. The *corona* $G \circ H$ of G and H is the graph obtained from the disjoint union of G and nH by

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joining the ith vertex of the copy of G to every vertex in the ith copy of H. Let $U \subset V(G)$. Then $G - U$ is a graph obtained from G by deleting all the vertices in $U \cap V(G)$ and their incident edges. By a subdivision of an edge uv we mean removing edge uv, adding a new vertex x, and adding edges ux and vx.

A subset $Q \subseteq V(G)$ is a *dominating set* of G if any vertex $y \in V(G) - Q$ is adjacent to at least one vertex $x \in Q$. We say that x dominates y in G or y is dominated by x in G. The concept of domination in graphs has existed in the literature for a long time. Recent articles on domination in graphs can be found in Haynes et al [3]. In this paper we study one of many variations of domination. A subset $S \subseteq V(G)$ is a *weak dominating set* of G if for any vertex $y \in V(G) - S$ there exists a vertex $x \in S$ adjacent to y in G and such that $\deg_G(x) \leq \deg_G(y)$. If G is a regular graph then any dominating set of G is also weak dominating set of G. A *minimal weak dominating set* S of a graph G is a weak dominating set that contains no weak dominating set of G as a proper subset. Weak dominating sets were considered in [4]. Let $d(G)$ and $w(G)$ denote the total number of dominating sets and weak dominating sets of a graph G, respectively.

Fact 1. (see [4]) *For $n \geq 2$, $d(P_{n+1}) = d(P_n) + d(P_{n-1}) + d(P_{n-2})$ with initial conditions $d(P_0) = 1$, $d(P_1) = 1$ and $d(P_2) = 3$.*

Fact 2. (see [4]) *For $n \geq 5$, $w(P_{n+1}) = w(P_n) + w(P_{n-1}) + w(P_{n-2})$ with initial conditions $w(P_3) = 2$, $w(P_4) = 4$ and $w(P_5) = 7$.*

2. Main Results

Proposition 3. *Let G be a connected graph on $n \geq 3$ vertices and let L be the set of pendant vertices of G. If S is any weak dominating set of G then $L \subseteq S$.*

Proof. Let S be a weak dominating set of G. On the contrary, suppose that there exists $z \in L$ such that $z \notin S$. Let x be the neighbor of z. Since $n \geq 3$, we see that $\deg_G(x) > \deg_G(z) = 1$. Hence z is not weakly dominated by x in G. Thus S is not a weak dominating set of G, a contradiction. □

Proposition 4. *Let G be a disconnected graph without any component being K_1 or K_2. Assume that L is the set of pendant vertices of G. If S is a weak dominating set of G then $L \subseteq S$.*

Given a tree T and an integer $k \geq 2$, a *pendant* (k-)*star* with k rays in T is a subgraph $K_{1,k}$ whose central vertex is of degree k in T, too, and is adjacent to $k-1$ or k leaves of T. Let $x \in V(T)$. Denote by $L(x)$ the set of leaves attached to the vertex x. Let $L[x] = L(x) \cup \{x\}$.

Observation 5. *Let T be a tree with $x \in V(T)$. Then for every $L'(x) \subset L(x)$, $w(T) = w(T - L'(x))$.*

Theorem 6. *Let T be an n-vertex tree with $n \geq 3$. Then $w(T) \geq 2$ with equality if and only if $T = K_{1,n-1}$.*

Proof. The inequality is obvious. It is easy to check that $w(K_{1,n-1}) = 2$. Assume that T is an n-vertex tree with $n \geq 3$ and $w(T) = 2$. We shall show that $T = K_{1,n-1}$. If $n = 3$ then $T = K_{1,2}$. Assume that $n \geq 4$ and $T \neq K_{1,n-1}$. This means that there exists a path of length at least 3 in T. Let $xyzu$ be a path in T with x, u being leaves. Assume that S is a weak dominating set of T. By Proposition 3, $\{x, u\} \subset S$. Thus, it follows that y is weakly dominated by x, and z is weakly dominated by u. Consequently, $\{x, y, u\} \subseteq S$ or $\{x, z, u\} \subseteq S$ or $\{x, y, z, u\} \subseteq S$ or $\{x, u\} \subseteq S$. This contradicts our assumption that $w(T) = 2$ and the proof is complete. □

Theorem 7. *For an arbitrary $n \geq 3$ there is no an n-vertex tree T with $w(T) = 3$.*

Theorem 8. *Let T be a tree with pendant star with k rays, $k \geq 3$. Assume that x is the central vertex of the pendant star and $L(x) = \{z_1, z_2, \ldots, z_p\}$, $p \geq 3$. Let T' be a tree obtained from $T - \{z_1\}$ by subdivision of the edge xz_2. Then $w(T') = 2w(T)$.*

Proof. Assume that S is a weak dominating set of T. Then, by Proposition 3, $\{z_1, \ldots, z_p\} \subseteq S$. Let $T^* = T - \{z_1\}$. By Observation 5, $w(T^*) = w(T)$. Let $\mathcal{S}_x$, $\mathcal{S}_{-x}$ denote the family of all weak dominating sets of T^* containing the vertex x, not containing x, respectively. Then $w(T^*) = |\mathcal{S}_x| + |\mathcal{S}_{-x}| = w(T)$. Assume that v is the neighbor of x and z_2 in T'. Let S' be a weak dominating set of T'. It is easily seen that $\{z_2, \ldots, z_p\} \subseteq S'$ and v is weakly dominated by z_2. Therefore either $x \in S'$ or $x \notin S'$. It is easy to check that every weak dominating set $S_1 \in \mathcal{S}_x$ and $S_2 \in \mathcal{S}_{-x}$ of T^* is a weak dominating set of T'. Moreover $S_1 \cup \{v\}$, $S_2 \cup \{v\}$ are weak dominating sets of T', too. Therefore, $w(T') = 2|\mathcal{S}_x| + 2|\mathcal{S}_{-x}| = 2w(T)$ and the proof is complete. □

Theorem 9. *Let T be an n-vertex tree, $n \geq 3$. Then for an arbitrary $m > n$ there is an m-vertex tree T^* such that $w(T^*) = w(T)$.*

Proof. Let $x \in V(T)$ and $L(x) \neq \emptyset$. We locally augment the tree T by adding to the vertex x the star $K_{1,p}$, $p = m - n$, so that x is identified with the center of the star $K_{1,p}$. Then we obtain tree T^* with $|V(T^*)| = m$. It is easily seen that $w(T^*) = w(T)$, which ends the proof. $\square$

For a given set $X \subset V(G)$, let $d_X(G)$ denote the number of all dominating sets in G with X included in each of the dominating sets. If $X = \{x\}$ then we write $d_x(G)$.

Fact 10. (see [1]) *Let z_1, z_2 be endvertices of P_n. Then for $n \geq 3$:*

a) $d_{z_1}(P_n) = d_{z_1}(P_{n-1}) + d_{z_1}(P_{n-2}) + d_{z_1}(P_{n-3})$ with initial conditions $d_{z_1}(P_i) = i$ for $i = 0, 1, 2$,

b) $d_{\{z_1,z_2\}}(P_n) = d_{\{z_1,z_2\}}(P_{n-1}) + d_{\{z_1,z_2\}}(P_{n-2}) + d_{\{z_1,z_2\}}(P_{n-3})$ with initial conditions $d_{\{z_1,z_2\}}(P_0) = 0$, $d_{\{z_1,z_2\}}(P_i) = 1$ for $i = 1, 2$.

Fact 11. (see [1]) *Let z_1, z_2 be endvertices of P_n. Then for $n \geq 3$:*

a) $d_{z_1}(P_n) = \frac{1}{2}(d(P_n) + d(P_{n-2}))$,

b) $d_{z_1}(P_n) = d_{\{z_1,z_2\}}(P_n) + d_{\{z_1,z_2\}}(P_{n-1})$,

c) $d_{\{z_1,z_2\}}(P_n) = \frac{1}{2}(d_{z_1}(P_{n+1}) - d_{z_2}(P_{n-1}))$.

On applying Fact 11 we have.

Proposition 12. *For $n \geq 3$, $w(P_n) = \frac{1}{4}(d(P_{n+1}) - d(P_{n-3}))$.*

Fact 13. (see [1]) *Let $n \geq 5$, $m \geq 3$ be integers. Then $d(P_{n,m}) = 2^{m-1} d_{z_1}(P_{n-m+1}) + d(P_{n-m})$.*

Theorem 14. *Let $n \geq 5$ and $3 \leq m < n - 1$. Then $w(P_{n,m}) = d(P_{n-m}) + d(P_{n-m-2})$.*

Proof. Let x_k be the vertex of degree m in the palm $P_{n,m}$ and let $L(x_k) = \{z_{k+1}, \ldots, z_n\}$. Let x_1 be the only leaf of $P_{n,m}$ not attached to x_k. Assume that S is a weak dominating set of $P_{n,m}$. By Proposition 3, $(L(x_k) \cup \{x_1\}) \subset S$. Consider the following cases.

1. $x_k, x_{k-1} \in S$. Let $\mathcal{S}$ be a family of all weak dominating sets S of $P_{n,m}$ such that $x_k, x_{k-1} \in S$. Denote $T' = P_{n,m} - L[x_k]$. Of course, $T' \approx P_{n-m}$. Since $x_1, x_k, x_{k-1} \in S$ and all remaining vertices of T' are of degree 2 in T', we have that every dominating set of T' containing vertices x_1, x_{k-1} is a weak dominating set of T'. Therefore, $S = S' \cup L[x_k]$ where S' is a dominating set of T' containing endvertices x_1, x_{k-1}. Thus, $|\mathcal{S}| = d_{\{x_1,x_{k-1}\}}(P_{n-m})$.

2. $x_k \in S$ and $x_{k-1} \notin S$. Let $\mathcal{S}'$ be a family of all weak dominating sets S of $P_{n,m}$ such that $x_k \in S$ and $x_{k-1} \notin S$. Thus, $x_{k-2} \in S$. Let $T'' = P_{n,m} - L[x_k] - \{x_{k-1}\}$. Then $T'' \approx P_{n-m-1}$. Analysis similar to that in case

1 shows that $S = S'' \cup L[x_k]$ where S'' is a dominating set of T'' containing endvertices x_1, x_{k-2}. Thus, $|\mathcal{S}'| = d_{\{x_1,x_{k-2}\}}(P_{n-m-1})$.

3. $x_k \notin S$. Let $\mathcal{S}''$ be a family of all weak dominating sets S of $P_{n,m}$ such that $x_k \notin S$. It is easily seen that x_k is weakly dominated by vertices $z_{k+1}, \ldots, z_n$. This follows by the same method as in cases 1-2, $|\mathcal{S}''| = d_{\{x_1,x_{k-1}\}}(P_{n-m}) + d_{\{x_1,x_{k-2}\}}(P_{n-m-1})$.

Consequently, $w(P_{n,m}) = 2(d_{\{x_1,x_{k-1}\}}(P_{n-m}) + d_{\{x_1,x_{k-2}\}}(P_{n-m-1}))$. By Fact 11, $w(P_{n,m}) = 2d_{x_1}(P_{n-m}) = d(P_{n-m}) + d(P_{n-m-2})$. □

Theorem 15. *Let G be a connected graph on n vertices, $n \geq 2$. Then for $p \geq 1$, $w(G \circ \overline{K_p}) = 2^n$.*

Proof. Assume that S is a weak dominating set of $G \circ \overline{K_p}$. Let $V(G) = \{x_1, \ldots, x_n\}$, $n \geq 2$. Denote by L the set of leaves of $G \circ \overline{K_p}$. Then, by Proposition 3, $L \subseteq S$. It follows that every vertex x_i, $i = 1, \ldots, n$, is weakly dominated by some elements of L. Thus, $S = S' \cup L$ where $S' \subset V(G)$. Hence $w(G \circ \overline{K_p}) = 2^n$. □

Denote by $\omega(G)$ the number of all minimal weak dominating sets of a graph G. It is easily seen that $\omega(P_1) = 1$, $\omega(P_2) = 2$. We consider path P_n with $n \geq 3$.

Theorem 16. *Let $n \geq 3$ be an integer. Then for $n \geq 11$*

$$\omega(P_n) = \omega(P_{n-3}) + 2\omega(P_{n-4}) + \omega(P_{n-5}) - \omega(P_{n-8})$$

with initial conditions $\omega(P_3) = \omega(P_4) = 1$, $\omega(P_5) = \omega(P_6) = 3$, $\omega(P_7) = 4$, $\omega(P_8) = 6$, $\omega(P_9) = 9$, $\omega(P_{10}) = 12$.

Proof. The initial conditions are obvious. Let $n \geq 11$. Assume that vertices of P_n are numbered in the natural fashion and S is a minimal weak dominating set of P_n. By Proposition 3, $x_1, x_n \in S$. We use the following notation:

$\mathcal{W}(P_n) = \{S \subseteq V(P_n) : S$ is a minimal weak dominating set of $P_n\}$,

$\mathcal{W}_1(P_n) = \{S \in \mathcal{W}(P_n) : x_{n-1} \in S\}$, $\mathcal{W}_2(P_n) = \mathcal{W}(P_n) - \mathcal{W}_1(P_n)$.

Let $\omega(P_n)$, $\omega_1(P_n)$, $\omega_2(P_n)$ stand for the cardinality of $\mathcal{W}(P_n)$, $\mathcal{W}_1(P_n)$, $\mathcal{W}_2(P_n)$, respectively. Then $\omega(P_n) = \omega_1(P_n) + \omega_2(P_n)$.

Consider the following cases.

Case 1. $x_{n-1} \in S$. Then, by definition of minimal weak dominating set, $x_{n-2}, x_{n-3} \notin S$ and $x_{n-4} \in S$ and $S - \{x_n, x_{n-1}\} \in \mathcal{W}(P_{n-4})$.

Case 2. $x_{n-1} \notin S$. Consider two subcases.

2.1. $x_{n-2} \notin S$. Then $x_{n-3} \in S$ and $S - \{x_n\} \in \mathcal{W}(P_{n-3})$.

2.2. $x_{n-2} \in S$. Then $x_{n-3} \notin S$. Consider two possibilities:

2.2.1. $x_{n-4} \in S$. Then $x_{n-5} \notin S$ and $S - \{x_n, x_{n-2}\} \in \mathcal{W}_2(P_{n-4})$.

2.2.2. $x_{n-4} \notin S$. Then $x_{n-5} \in S$ and $S - \{x_n, x_{n-2}\} \in \mathcal{W}(P_{n-5})$.

Therefore $\omega(P_n) = \omega(P_{n-4}) + \omega_2(P_n)$ and $\omega_2(P_n) = \omega(P_{n-3}) + \omega_2(P_{n-4}) + \omega(P_{n-5})$. Hence $\omega(P_n) - \omega(P_{n-4}) = \omega(P_{n-3}) + \omega_2(P_{n-4}) + \omega(P_{n-5})$. Thus, $\omega_2(P_n) = \omega(P_{n+4}) - \omega(P_n) - \omega(P_{n-1}) - \omega(P_{n+1})$. Consequently, $\omega(P_n) = \omega(P_{n-4}) + \omega(P_{n+4}) - \omega(P_n) - \omega(P_{n-1}) - \omega(P_{n+1})$. Therefore for $n \geq 11$, $\omega(P_n) = \omega(P_{n-3}) + 2\omega(P_{n-4}) + \omega(P_{n-5}) - \omega(P_{n-8})$, which ends the proof. $\square$

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