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SEQUENCES FROM CENTERED PENTAGONS OF INTEGERS

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Abstract: This paper presents a number of sequences based on integers arranged in a centered pentagon structure. This approach provides a simple derivation of some well known sequences. In addition, a number of new integer sequences are obtained.

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1. Introduction

In a previous paper [1], several well-known sequences (and many new sequences), were derived from two-dimensional arrays of integers. These arrays were in the form of triangles, squares and hexagons. In this paper, we consider arrays of integers in a pentagonal (five-sided) shape.

A figurate number is a number that can be represented as a regular geometric pattern. The centered pentagonal numbers are a class of figurate numbers. The figures are formed by a central element, surrounded by concentric pentagons. Each side of a ring contains one more element than a side in the previous ring. Here the elements are consecutive integers, starting with 1 in the centre and increasing along each ring. Thus the first ring from the centre has $2, 3, 4, 5, 6$, and these numbers will be used to denote the vertices of the array.

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An array with n rings contains

$$s_n = \frac{5n^2 + 5n + 2}{2}$$

elements. The first numbers in the corresponding sequence (starting from $n = 0$), are

$$1, 6, 16, 31, 51, 76, \ldots$$

which is sequence A005891 in the Encyclopedia of Integer Sequences maintained by Sloane [2]. The generating function of these centered pentagonal numbers is

$$\frac{(x^2 + 3x + 1)}{(1 - x)^3} = 1 + 6x + 16x^2 + 31x^3 + \ldots .$$

This sequence represents the maximum integer in the n-th ring of the array.

2. Pentagonal Arrays of Numbers

Starting at the k-th vertex, one can form five sequences of integers given by

$$s_n = 2 + \sum_{i=1}^{n} 5i + (k - 7)n + 2 = \frac{5n^2 + (2k - 9)n + 4)}{2} . \qquad (1)$$

For $k = 2$ to 6 the sequences are

$$\begin{array}{llllll} 2, & 7, & 17, & 32, & 52, & \ldots \\ 3, & 9, & 20, & 36, & 57, & \ldots \\ 4, & 11, & 23, & 40, & 62, & \ldots \\ 5, & 13, & 26, & 44, & 67, & \ldots \\ 6, & 15, & 29, & 48, & 72, & \ldots \end{array}$$

All these sequences are new. Taking the sum of the integers starting from each vertex gives

$$s_l = \sum_{n=1}^{l} \frac{5n^2 + (2k - 9)n + 4)}{2} = \frac{l(5l^2 + (3k - 6)l + 3k + 1)}{6} . \qquad (2)$$

For $k = 2$ to 6 the sequences are

$$\begin{array}{llllll} 2, & 9, & 26, & 58, & 110, & \ldots \\ 3, & 12, & 32, & 68, & 125, & \ldots \\ 4, & 15, & 38, & 78, & 140, & \ldots \\ 5, & 18, & 44, & 88, & 155, & \ldots \\ 6, & 21, & 50, & 98, & 170, & \ldots \end{array}$$

The second sequence is A037236 in [2], but it is listed as the expansion of $(3 + 2x^2)/(1 - x)^4$ with no interpretation. The four other sequences, as well as the reminder in this paper, are new.

Summing the elements in the n-th ring of the array gives

$$s_n = \frac{5n(5n^2+3)}{2} \tag{3}$$

which has terms

$$20, \quad 115, \quad 360, \quad 830, \quad 1600, \quad \ldots .$$

Note that the ones digit follows the pattern $0, 5, 0, 0, 0, 5, 0, 0, 0, 5, \ldots$ This is easily proven. For $n = 0 \bmod 4$, the result is obvious. For $n = 1 \bmod 2$, substituting $n = 2x + 1$ gives

$$100x^3 + 150x^2 + 90x + 20$$

and finally for $n = 2$ mod 4, we have

$$800x^3 + 1200x^2 + 630x + 115\,.$$

The sum of the elements in all the rings (including ring 0) is then

$$\begin{aligned} 1 + \sum_{n=1}^{l} \frac{5n(5n^2+3)}{2} = 1 + \frac{5l(l+1)(5l^2+5l+6)}{8} \\ = \frac{(5l^2+5l+4)(5l^2+5l+2)}{8} \end{aligned} \tag{4}$$

which has terms

$$1, \quad 21, \quad 136, \quad 496, \quad 1326, \quad \ldots .$$

One can also consider wedges in the arrays. For example, the sequence for the wedge between vertices 2 and 3 is $2+3 = 5, 7+8+9 = 24, 17+18+19+20 = 74, \ldots$. The sequences in increasing wedge size of 2 to k are generated by

$$s_n = \frac{((k-2)n+1)(5n^2+(k-7)n+4)}{2}\,. \tag{5}$$

For $k = 3$ to 6 these sequences are

$$\begin{array}{rrrrrr} 5, & 24, & 74, & 170, & 327, & \ldots \\ 9, & 45, & 140, & 324, & 627, & \ldots \\ 14, & 70, & 215, & 494, & 952, & \ldots \\ 20, & 99, & 299, & 680, & 1302, & \ldots \end{array}$$

Since the last sequence represents just four wedges of the array, there is a difference between it and (3). For the first ring, they are the same, while the difference for the second ring is 16 (just the integer between vertices 2 and 6 in the ring). The sequence for the differences is

$$\frac{(n-1)(5n^2+4n+4)}{2} \tag{6}$$

which gives

$$0, \quad 16, \quad 61, \quad 150, \quad 298, \quad \ldots .$$

Taking the sums of the wedge elements gives

$$\begin{aligned} s_l &= \sum_{n=1}^{l} \frac{((k-2)n+1)(5n^2+(k-7)n+4)}{2} \\ &= \frac{l}{24}(12k - 6l - 9kl + 16l^2 + 4k^2l^2 + 6k^2l - 6kl^2 + 15kl^3 - 30l^3 - 4). \end{aligned} \quad (7)$$

For $k = 3$ to 6 the sequences are

$$\begin{array}{cccccc} 5, & 29, & 103, & 273, & 600, & \ldots \\ 9, & 54, & 194, & 518, & 1145, & \ldots \\ 14, & 84, & 299, & 793, & 1745, & \ldots \\ 20, & 119, & 418, & 1098, & 2400, & \ldots \end{array}$$

The sequence for the sum of the differences (6) is

$$s_l = \sum_{n=1}^{l} \frac{(n-1)(5n^2+4n+4}{2} = \frac{l(l-1)(15n^2+41l+50}{2} \quad (8)$$

which gives

$$0, \quad 16, \quad 77, \quad 227, \quad 525, \quad \ldots .$$

One can also form sequences from other wedges in the arrays. Between vertices 3 and 4 we have

$$s_n = \frac{(n+1)(5n^2-2n+4)}{2} \quad (9)$$

which gives

$$7, \quad 30, \quad 86, \quad 190, \quad 357, \quad \ldots .$$

Between vertices 4 and 5 we have

$$s_n = \frac{(n+1)(5n^2+4)}{2}, \quad (10)$$

which gives

$$9, \quad 36, \quad 98, \quad 210, \quad 387, \quad \ldots .$$

From (9) and (10), in general, the sequences for the wedges between adjacent vertices k and $k+1$ is

$$s_n = \frac{(n+1)(5n^2+2(k-4)n+4)}{2}. \quad (11)$$

For $k = 5$, this gives the sequence

$$11, \quad 42, \quad 110, \quad 230, \quad 417, \quad \ldots$$

Taking the sums of the elements in the wedges between adjacent vertices gives

$$s_l = \sum_{n=1}^{l} \frac{(n+1)(5n^2 + 2(k-4)n + 4)}{2}$$
$$= \frac{l}{24}(15l^3 + 18l^2 + 8kl^2 - 27l + 24kl + 16k + 18)\,. \quad (12)$$

For $k = 3$ to 5 the sequences are

$$\begin{array}{cccccc} 7, & 37, & 123, & 313, & 670, & \ldots \\ 9, & 45, & 143, & 353, & 740, & \ldots \\ 11, & 53, & 163, & 393, & 810, & \ldots \end{array}$$

The sequence for $k = 2$ was given previously.

For pairs of wedges, the sequences starting at vertex k are given by

$$s_n = \frac{(2n+1)(5n^2 + (2k-7)n + 4)}{2}\,. \quad (13)$$

For $k = 2$, the sequence was given previously. For $k = 3$ and 4, we have

$$\begin{array}{cccccc} 12, & 55, & 161, & 360, & 682, & \ldots \\ 15, & 65, & 182, & 396, & 737, & \ldots \end{array}$$

Taking the sums of the elements in the wedge pairs between adjacent vertices gives

$$s_l = \sum_{n=1}^{l} \frac{(2n+1)(5n^2 + (2k-7)n + 4)}{2}$$
$$= \frac{l}{12}(15l^3 + 12l^2 + 8kl^2 + 18kl - 9l + 10k + 18)\,. \quad (14)$$

For $k = 3$ to 5 the sequences are

$$\begin{array}{cccccc} 12, & 67, & 228, & 588, & 1270, & \ldots \\ 15, & 80, & 262, & 658, & 1395, & \ldots \\ 18, & 93, & 296, & 728, & 1520, & \ldots \end{array}$$

The sequence for $k = 2$ was given previously.

Sequence (6) gives the values between vertices 2 and 6. The sequences between other pairs of vertices are generated by

$$s_n = \frac{(n-1)(5n^2 + 2(k-4)n + 4)}{2}\,. \quad (15)$$

For $k = 6$, we obtain (6). For $k = 2, 3, 4$ and 5, we have

$$\begin{array}{cccccc} 0, & 8, & 37, & 102, & 218, & \ldots \\ 0, & 10, & 43, & 114, & 238, & \ldots \\ 0, & 12, & 49, & 126, & 258, & \ldots \\ 0, & 14, & 55, & 138, & 278, & \ldots \end{array}$$

Taking the sums of the elements between vertices gives

$$s_l = \sum_{n=1}^{l} \frac{(n-1)(5n^2 + 2(k-4)n + 4)}{2}$$
$$= \frac{l}{24}(l-1)(15l^2 + 8kl - 7l + 8k + 2)\,. \quad (16)$$

For $k = 2$ to 5 the sequences are

$$\begin{array}{cccccc} 0, & 8, & 45, & 147, & 365, & \ldots \\ 0, & 10, & 53, & 167, & 405, & \ldots \\ 0, & 12, & 61, & 187, & 445, & \ldots \\ 0, & 14, & 69, & 207, & 485, & \ldots \end{array}$$

The sequence for $k = 6$ was given previously.

References

[1] T.A. Gulliver, Sequences from arrays of integers, *Int. Math. Journal*, **1** (2002), 323-332.

[2] N.J.A. Sloane, *On-Line Encyclopedia of Integer Sequences*, http://www.research.att.com/~njas/sequences/index.html.