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A COMPUTATIONAL STUDY ON THE NONLINEAR HARDENING CURVED BEAM PROBLEM

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Abstract: A computational model is developed to estimate the partially plastic stress state in a slender curved beam subjected to couples at its end sections. The model proposed is based on the von Mises' yield criterion, the total deformation theory, and nonlinear strain hardening material behavior. A state of plane stress and infinitesimal deformations are presumed. Using formal nondimensional variables and an appropriate stress function, a single second order nonlinear differential equation describing the deformation behavior of the curved beam is obtained. A shooting technique using Newton iterations with numerically approximated tangents is used for the numerical integration of the governing equation. The stress responses of the beam are computed in purely elastic and partially plastic stress states and the results are presented in graphical forms.

AMS Subject Classification: 26A33
Key Words: curved beam, stress analysis, elastoplasticitiy, von Mises' criterion, nonlinear hardening

1. Definitions and Notations

a, b : inner and outer radii of the curved beam.

C_i, A, ϕ : integration constants ($i = 1, 2$).

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H, m : material hardening parameters.

E : modulus of elasticity.

M : bending moment (dimensionless form $\overline{M} = M/(\sigma_0 b^2 t_z)$).

r, θ : cylindrical polar coordinates (dimensionless radial coordinate $\overline{r} = r/b$).

r_{NA} : radial coordinate of the neutral axis.

r_{EP} : radial coordinate of elastic-plastic interface.

t_r, t_z : thickness in r and z directions of the beam, respectively.

u : radial displacement component (dimensionless form $\overline{u} = uE/(\sigma_0 b)$).

v : tangential displacement component (dimensionless form $\overline{v} = vE/(\sigma_0 b)$).

Y : stress function.

ν : Poisson's ratio.

$\gamma_{r\theta}$: shear strain component (normalized form $\overline{\gamma}_{r\theta} = \gamma_{r\theta} E/\sigma_0$).

ϵ_j : normal strain component in $j-$direction (normalized form $\overline{\epsilon}_j = \epsilon_j E/\sigma_0$).

ϵ_{EQ} : equivalent plastic strain.

σ_0, σ_y : initial and subsequent yield stress, respectively.

σ_j : normal stress component in $j-$direction (dimensionless form $\overline{\sigma}_j = \sigma_j/\sigma_0$).

2. Introduction

The articles authored by Shaffer and House Jr. in the late 1950s [7], [8], [9] may be considered as the pioneering ones not only in defining and formulating but also in deriving a partially plastic solution to the curved beam problem. In their theoretical studies, Shaffer and House Jr. considered a narrow rectangular cross section wide curved beam subjected to couples at its end sections. The geometry and the loading of the beam as well as the coordinate system considered therein is as shown in Figure 1. The beam deforms and stresses develop as the bending moment M is increased slowly in the direction to straighten it. Initially the elastically deforming beam begins to plasticize at the inner surface $r = a$ as soon as the moment reaches a critical value M_E called the elastic limit bending moment. As the plastic region formed around the concave surface spreads into the beam with increasing M values, another plastic region forms at the outer surface, i.e. at $r = b$. Thereafter, the beam is composed of an inner plastic, an elastic and an outer plastic region. Shaffer and House Jr. [7], [9] derived a plane strain analytical solution to this elastic-plastic deformation problem for an ideally plastic material using Tresca's yield criterion. Following in time, the solutions in the plastic state of stress for ideally plastic materials by Hill [6], and for strain hardening materials by Dadras and Majlessi [3], and a comprehensive retreatment of the problem in the elastic state of stress by Timoshenko and Goodier [10] appeared in the literature. Recently, Dadras [2] has obtained an analytical solution of elastoplastic pure bending of a linear strain hardening curved beam under plane strain supposition. Like the solution of Shaffer and House Jr. [7], this work was based on Tresca's yield criterion.

In the present work, we extend earlier studies to include nonlinear strain hardening and the use of von Mises' yield criterion, which is known to comply better with experimental observations. A computational model is developed for this purpose. The model is based on the von Mises' yield criterion, the deformation theory of plasticity, and a Swift-type nonlinear strain hardening law. In contrast to the work of Dadras [2], a state of plane stress as in [10] is assumed. Using nondimensional variables and an appropriate stress function, a single second order nonlinear differential equation describing the elastoplastic response of the curved beam is obtained. A shooting technique using Newton iterations with numerically approximated tangents is used for the numerical integration of the governing equation. Sample computations are carried out and critical values of the parameters like neutral axis, and elastic-plastic border radius are determined.

3. The Computational Model

The notation of Timoshenko and Goodier [10] is used. However, the derivation of the governing equation is performed in terms of formal nondimensional and normalized variables for computational purposes. These variables are, radial coordinate: $\overline{r} = r/b$, normal stress: $\overline{\sigma}_j = \sigma_j/\sigma_0$, strains: $\overline{\epsilon}_j = \epsilon_j E/\sigma_0$, and $\overline{\gamma}_{r\theta} = \gamma_{r\theta} E/\sigma_0$, displacements: $\overline{u} = uE/(\sigma_0 b)$, and $\overline{v} = vE/(\sigma_0 b)$, and bending moment: $\overline{M} = M/(\sigma_0 b^2 t_z)$, with b being the outer radius, σ_0 the yield strength, E the modulus of elasticity, and t_z the thickness (see Figure 1). The equations given below are written in terms of these variables, but to simplify the notation overbars are dropped.

A state of plane stress, i.e. $\sigma_z = 0$, and small deformations are assumed.

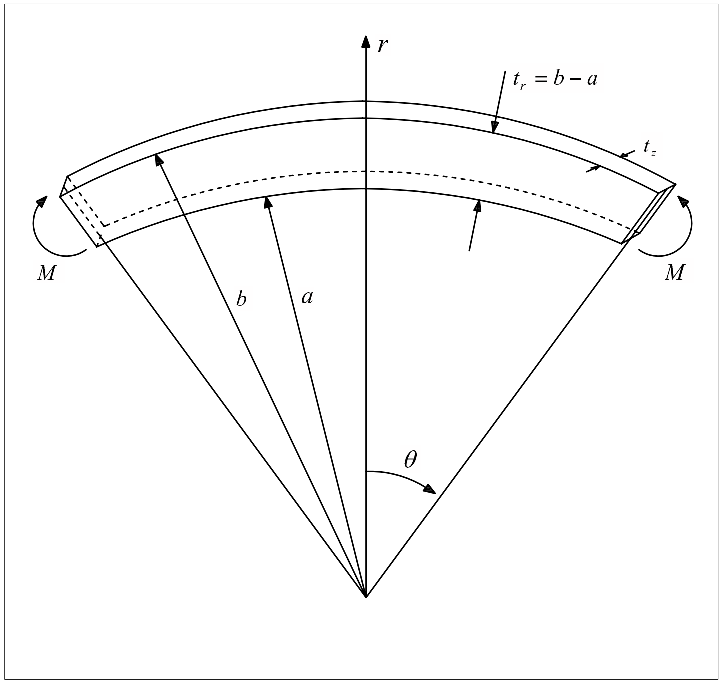

Figure 1: The geometry of the curved beam and the coordinate system used

The strain-displacement relations

$$\epsilon_r = \frac{\partial u}{\partial r} \ ; \ \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \ ; \ \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r} = 0, \tag{1}$$

the equation of equilibrium

$$\sigma_\theta = \frac{\partial}{\partial r}\left(r\sigma_r\right), \tag{2}$$

the compatibility relation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial \epsilon_\theta}{\partial r}\right) - \frac{\partial \epsilon_r}{\partial r} = 0, \tag{3}$$

and the equations of generalized Hooke's law

$$\epsilon_r = \epsilon_r^p + \sigma_r - \nu\sigma_\theta, \tag{4}$$

$$\epsilon_\theta = \epsilon_\theta^p + \sigma_\theta - \nu\sigma_r, \tag{5}$$

$$\epsilon_z = \epsilon_z^p - \nu(\sigma_r + \sigma_\theta), \tag{6}$$

form the basis for the entire analysis. In above ν is the Poisson's ratio and the notation ϵ_j^p has been used to denote a plastic strain component. Furthermore,

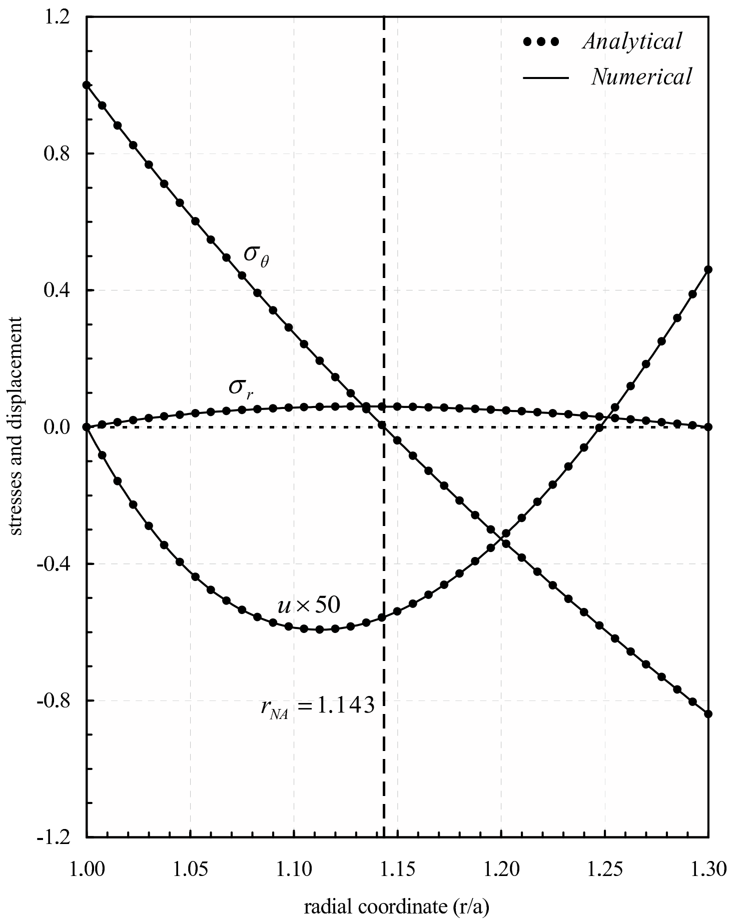

Figure 2: Comparison of numerical (solid lines) and analytical (dots) stresses and displacement in a beam of $b/a = 1.3$ at the elastic limit load $M_E = 8.09989 \times 10^{-3}$

a straight forward manipulation on strain-displacement relations, equation (1), and the compatibility, equation (3), lead to

$$v = \phi r \theta + A \sin\theta, \tag{7}$$

$$u = r\epsilon_\theta - \phi r - A\cos\theta, \tag{8}$$

$$\frac{\partial}{\partial r}\left(r\epsilon_\theta\right) - \epsilon_r = \phi, \tag{9}$$

where ϕ and A represent two arbitrary constants to be determined.

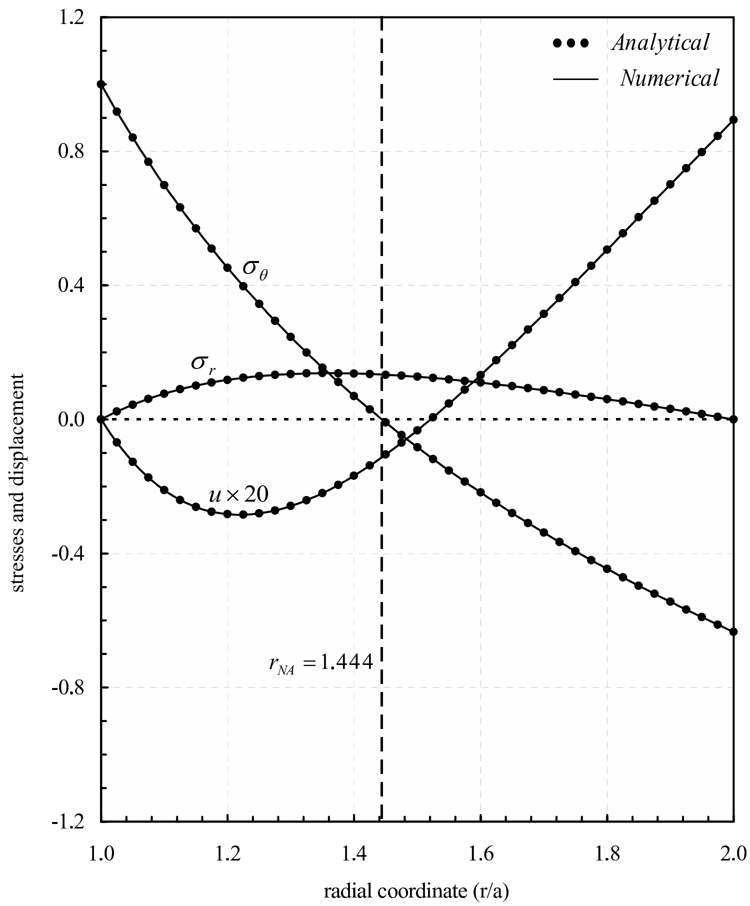

Figure 3: Comparison of numerical (solid lines) and analytical (dots) stresses and displacement in a beam of $b/a = 2$ at the elastic limit load $M_E = 3.22363 \times 10^{-2}$

Let the stress function Y be defined as $Y = r\sigma_r$, so that from the equation of equilibrium $\sigma_\theta = Y'$. We put Y in equations (4)-(5) and then substitute the resulting expressions in the compatibility (9) to obtain

$$r^2 \frac{d^2Y}{dr^2} + r\frac{dY}{dr} - Y = r\left[\phi + \epsilon_r^p - \epsilon_\theta^p - r\frac{d\epsilon_\theta^p}{dr}\right]. \tag{10}$$

This is the governing differential equation for the analysis of partially plastic curved beam. It is solved subjected to the conditions

$$Y(a) = Y(1) = 0\ ;\ \int_a^1 Y' r dr = -M. \tag{11}$$

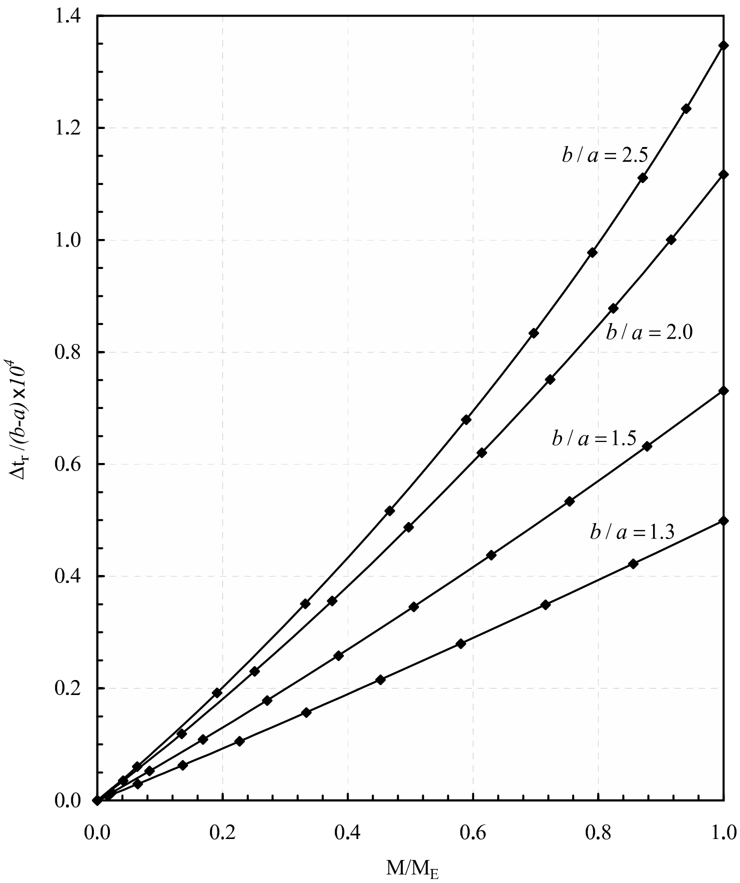

Figure 4: Relative change in thickness: $\Delta t_r/(b-a)$ as a function of M/M_E for different b/a ratios in the elastic range. Solid lines belong to exact calculation, diamonds to approximate ones

Note that at the elastic-plastic border the plastic strains ϵ_j^p, and their derivatives vanish and equation (10) reduces to the elastic equation

$$r^2\frac{d^2Y}{dr^2}+r\frac{dY}{dr}-Y=r\phi. \tag{12}$$

Hence, the continuity of the stress components and the displacements at the elastic-plastic border are automatically satisfied. However, the numerical solution of the governing equation, equation (10), is not possible unless explicit expressions for the plastic strains are substituted. This is done next as the final step to complete the formulation.

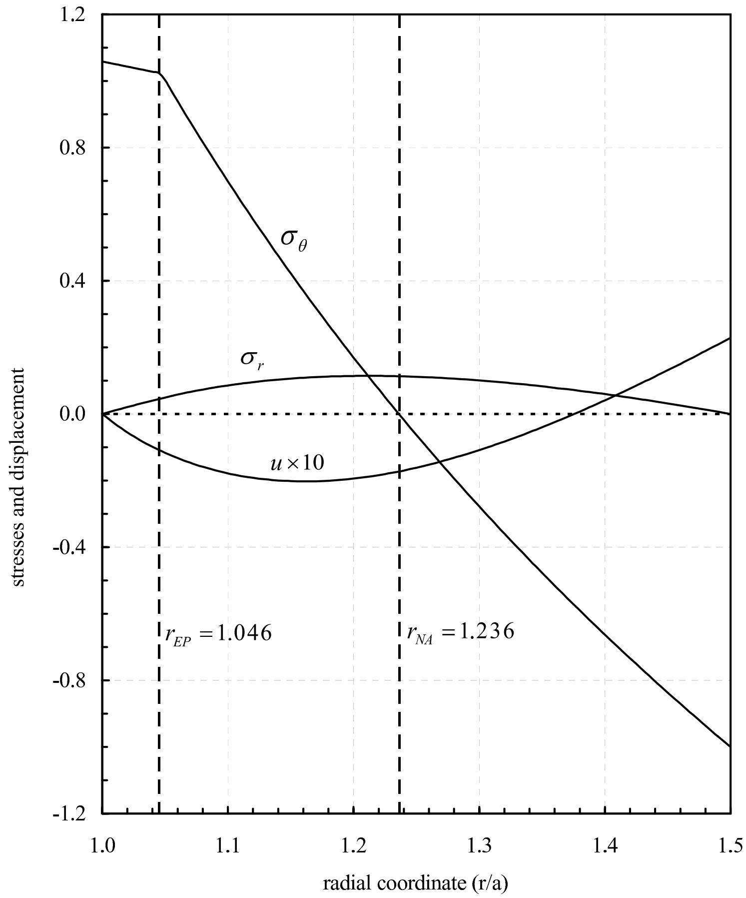

Figure 5: Stresses and displacement in a partially plastic curved beam of $b/a = 1.5$ under $M = 2.05763 \times 10^{-2}$

According to the total deformation theory, the plastic strains are given by

$$\epsilon_r^p = \frac{\epsilon_{EQ}}{\sigma_y}\left[\sigma_r - \frac{1}{2}\sigma_\theta\right] \;;\; \epsilon_\theta^p = \frac{\epsilon_{EQ}}{\sigma_y}\left[\sigma_\theta - \frac{1}{2}\sigma_r\right] \;;\; \epsilon_z^p = -(\epsilon_r^p + \epsilon_\theta^p), \tag{13}$$

where σ_y is the yield stress, and ϵ_{EQ} the equivalent plastic strain. On the other hand, for plane stress, the von Mises' criterion reads

$$\sigma_y = \sqrt{\sigma_r^2 - \sigma_r\sigma_\theta + \sigma_\theta^2}. \tag{14}$$

Using a Swift-type nonlinear hardening law, ϵ_{EQ} is related to σ_y via

$$\epsilon_{EQ} = \left[\sigma_y^m - 1\right]\frac{1}{H}, \tag{15}$$

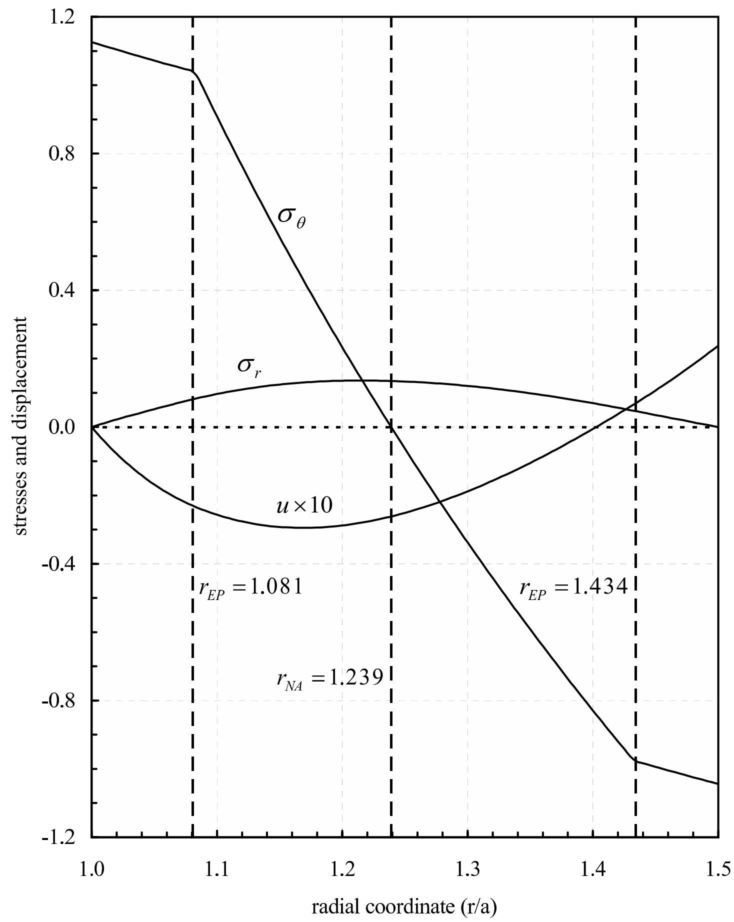

Figure 6: Stresses and displacement in a partially plastic curved beam of $b/a = 1.5$ under $M = 2.4 \times 10^{-2}$

where m and H are material parameters describing the hardening behavior of the beam. If the stresses in equations (13)-(15) are all expressed in terms of Y and Y', and substituted in equation (10) afterwards, an equation of the form

$$\frac{d^2Y}{dr^2} = F(r, Y, \frac{dY}{dr}), \tag{16}$$

is obtained. Making use of the fact that $\sigma_\theta = Y'$, equation (16) is transformed into an initial value problem:

$$\frac{dY}{dr} = \sigma_\theta, \tag{17}$$

$$\frac{d\sigma_\theta}{dr} = F(r, Y, \sigma_\theta), \tag{18}$$

in $a \leq r \leq 1$ subjected to

$$Y(a) = Y_a = 0 \; ; \; \sigma_\theta(a) = \left.\frac{dY}{dr}\right|_{r=a}. \tag{19}$$

The initial value Y_a is known; however, the gradient $dY/dr|_{r=a}$ is not known. This gradient can be computed iteratively using Newton's method accompanied by the boundary condition $Y(1) = 0$. Having X_{k-1} and Δ denote the value of $dY/dr|_{r=a}$ at iteration number $k-1$, and a small increment of the order 10^{-3}, respectively, we perform 3 runs in every iteration to generate the gradient in Newton's equation. At the $k-$th iteration we perform runs

1. starting with X_{k-1} to obtain $f_1 = Y(1)$,
2. starting with $X_{k-1} + \Delta$ to obtain $f_2 = Y(1)$,
3. starting with $X_{k-1} - \Delta$ to obtain $f_3 = Y(1)$.

A better approximation for $dY/dr|_{r=a}$ can now be obtained from

$$dY/dr|_{r=a} = X_k = X_{k-1} - \frac{2\Delta f_1}{f_2 - f_3}. \tag{20}$$

Iterations are repeated until $|X_k - X_{k-1}|$ is less than a specified error tolerance.

On the other hand, an outer iteration loop is performed to estimate the value of ϕ. An iteration scheme similar to that given above is constructed. At each main iteration, the problem is solved three times using ϕ^{k-1}, $\phi^{k-1}+\Delta$ and $\phi^{k-1} - \Delta$ respectively, and the corresponding integrals $\int_a^1 \sigma_\theta r dr$ are calculated. Aiming at $F(\phi) = \int_a^1 \sigma_\theta r dr + M = 0$, a better approximation ϕ^k is then obtained from

$$\phi^k = \phi^{k-1} - \frac{2\Delta\, F(\phi^{k-1})}{F(\phi^{k-1} + \Delta) - F(\phi^{k-1} - \Delta)}. \tag{21}$$

Starting with a reasonable initial estimate ϕ^0, this iteration scheme converges to the result with sufficient accuracy only in a few iterations. The result of a purely elastic analytical calculation, for example, provides a perfect initial estimate ϕ^0 to begin. The advantages of this procedure are the stability, the rate of convergence and the availability of state-of-the-art ODE solvers for accurate integrations [5], [4], [1].

4. Analytical Stresses in the Elastic State

These stresses are based on the analytical solution of equation (12), given by

$$Y = \frac{C_1}{r} + C_2 r + \frac{1}{2}\phi r \ln r. \tag{22}$$

The stresses and the radial displacement then become

$$\sigma_r = \frac{C_1}{r^2} + C_2 + \frac{1}{2}\phi \ln r, \tag{23}$$

$$\sigma_\theta = -\frac{C_1}{r^2} + C_2 + \frac{1}{2}\phi(1 + \ln r), \tag{24}$$

$$u = -\frac{1}{r}(1+\nu)C_1 + (1-\nu)C_2 r - \frac{1}{2}[1 - (1-\nu)\ln r]\phi r - A\cos\theta. \tag{25}$$

The conditions given by equation (11) are adjoined with the condition $u(a, 0) = 0$ to determine the unknowns. The result is

$$C_1 = \frac{4Ma^2 \ln a}{(1-a^2)^2 - 4a^2 \ln^2 a}, \tag{26}$$

$$C_2 = -\frac{4Ma^2 \ln a}{(1-a^2)^2 - 4a^2 \ln^2 a}, \tag{27}$$

$$\phi = -\frac{8M\left(1-a^2\right)}{(1-a^2)^2 - 4a^2 \ln^2 a}, \tag{28}$$

$$A = \frac{4Ma\left(1-a^2-2\ln a\right)}{(1-a^2)^2 - 4a^2 \ln^2 a}. \tag{29}$$

Yielding commences at the inner surface of the beam. Since the radial stress is zero at this radial location, the von Mises yield criterion, equation (14), reduces to $\sigma_\theta\,(a) = 1$. Solution of $\sigma_\theta\,(a) = 1$ for M brings the elastic limit into possession as

$$M_E = -\frac{\left(1-a^2\right)^2 - 4a^2 \ln^2 a}{4\left(1-a^2+2\ln a\right)}. \tag{30}$$

The neutral surface or axis r_{NA} of the beam is the radial location at which $\sigma_\theta\,(r_{NA}) = 0$. The use of equations (26)-(28), and the expression for σ_θ, i.e. equation (24), lead to the result

$$a^2 \ln a(1 + r_{NA}^2) + (1-a^2)(1 + \ln r_{NA})r_{NA}^2 = 0. \tag{31}$$

The neutral axis of the elastic curved beam is obtained by the numerical solution of this equation. The analytical expressions derived in this section will be used for the validation of the numerical solution procedure that follows.

5. Sample Calculations

The Poisson's ratio is taken as $\nu = 0.3$ throughout the calculations. First, a verification of the numerical solution procedure is performed. The slender beams with ratios $b/a = 1.3$, and $b/a = 2$ are considered. It is noted that these beams were analyzed in the elastic state of stress by Timoshenko and Goodier [10] as well. Thus, the results given here may be crosschecked by using [10]. The elastic limit couple moment for the beam of $b/a = 1.3$ is calculated from equation (30) as $M_E = 8.09989 \times 10^{-3}$. For $M = 8.09989 \times 10^{-3}$ five main iterations are performed to determine the constant ϕ, which converges to -7.01251. On the other hand, under $M_E = 8.09989 \times 10^{-3}$, from equations (26)-(29), the elastic constants are determined as $C_1 = -1.33321$, $C_2 = 1.33321$, $\phi = -7.01251$, and $A = 6.16347$. In addition, the solution of the exact expression given by equation (31) for the neutral axis gives $r_{NA}/a = 1.143$, which agrees very well with that obtained by the shooting solution. The numerical (solid lines) and analytical (dots) stresses and the radial displacement are plotted in Figure 2. Perfect agreement between numerical and exact solutions is obtained. For the beam with $b/a = 2$, the unknowns turn out to be $M_E = 3.22363 \times 10^{-2}$, $C_1 = -0.272337$, $C_2 = 0.272337$, $\phi = -2.35740$, $A = 1.67870$, and $r_{NA}/a = 1.444$. The distributions of the response variables in the beam are shown in Figure 3. As in Figure 2, the solid lines belong to numerical and the dots to analytical solutions. Again, perfect agreement is obtained. These calculations show that the numerical solution algorithm performs well and the computer program that implements this algorithm functions properly.

The change in dimensions of the beam as it deforms is of engineering interest as well. Most important of all, the thickness $t_r = b - a$ under pure bending of the curved beam is determined from

$$t_r = t_{r,0} - [u(a,\theta) - u(1,\theta)], \tag{32}$$

where $t_{r,0}$ is the thickness when the beam is stress free, see [2]. At the plane of symmetry, i.e. $\theta = 0$, $u(a,\theta) = 0$ since the beam is rigidly fixed at this point. Hence, by using equations (25)-(29), at $\theta = 0$, and introducing the ratio $\mathbf{f} = \sigma_0/E$, we come up with

$$t_r = t_{r,0} + \frac{4M\mathbf{f}(1-a)}{1 - a^2 - 2a\ln a}, \tag{33}$$

in the elastic range. However, the use of this equation to estimate t_r by simply substituting M, $\mathbf{f}$ and b/a is only approximate since the thickness changes as M increases, which then gives rise to a change in the ratio b/a. Therefore, to do an exact calculation, convergence ought to be sought as both M and b/a

are allowed to change very slowly. The relative change in thickness: $\Delta t_r/(b-a) = (t_r - t_{r,0})/(b-a)$ for a bar made of ASTM-A36 steel ($\mathbf{f} = 0.00125$) is calculated using b/a as a parameter, and the results are plotted in Figure 4. In this figure, solid lines show the results of exact calculations and diamonds approximate ones. It is to be noted that these results are in well agreement with those obtained by Shaffer and House Jr. [9]. Two important points should be brought into immediate attention. First, the relative change in thickness for all b/a ratios is negligibly small in accordance with the small deformation theory. Second, as the natural outcome of the first, the results of exact and approximate calculations concur perfectly. Moreover, as observed in Figure 4, the relative change in thickness increases almost linearly in the elastic range $0 \leq M \leq M_E$. Shaffer and House Jr. [9] point out that for a perfectly plastic material, Δt_r goes on increasing slowly in the elastic-plastic range, reaching to a maximum point and then decreasing to zero when the beam turns into fully plastic. However, in all stages of partially plastic deformation, Δt_r is so small that it can be neglected without loss of accuracy [3].

The beam of $b/a = 1.5$ undergoes plastic deformation at $r = a$ as soon as M reaches the limit $M_E = 1.60273 \times 10^{-2}$. For couples $M \geq M_E$, the beam becomes partially plastic. The plastic region formed at $r = a$ spreads into the beam with increasing values of M. As M is further increased, another limit $M = M_I$ is reached under which another plastic region begins to form at $r = 1$. Thereafter, the beam is composed of an inner plastic region, an elastic region and an outer plastic region. Using the hardening parameters $m = 1.2$ and $H = 0.25$, the limit M_I for the beam of $b/a = 1.5$ is determined as $M_I = 2.05763 \times 10^{-2}$. Figure 5 shows the consequent distributions of the response variables. The elastic plastic border radius and the neutral axis are computed as $r_{EP}/a = 1.046$ and $r_{NA}/a = 1.236$, respectively. The beam is composed of an inner plastic region in $1 \leq r/a \leq 1.046$, and an elastic region in $1.046 \leq r/a \leq 1.5$. As a final example, we take $M = 2.4 \times 10^{-2} > M_I$ and compute the stresses. The results are plotted in Figure 6. As seen in this figure, the curved beam is composed of an inner plastic region in $1 \leq r/a \leq 1.081$, an elastic region in $1.081 \leq r/a \leq 1.434$, and an outer plastic region in $1.434 \leq r/a \leq 1.5$. The neutral axis, on the other hand, is located at $r_{NA} = 1.239$.

6. Conclusion

A unified treatment of the boundary value problem of partially plastic curved beam is presented using the von Mises' yield criterion, the deformation theory of plasticity and general nonlinear strain hardening material behavior. It is well known that the von Mises' yield criterion leads to estimations that act in accordance with experimental observations. A Swift-type hardening law is used not only for its simple form but also for its ability to well describe nonlinear hardening material behaviour. However, the model is designed in such a way that any other hardening law or polynomial strain-yield stress relation may be incorporated. The approach developed here can easily be adapted to the analysis of numerous plane stress and plane strain applications of engineering interest.

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