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# GLOBAL ATTRACTORS OF STRONG SOLUTION FOR THE KOLMOGOROV-SPIEQEL-SIVASHINSKY EQUATION

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**Abstract:** Using a method based on the non-compactness measure theory, we prove the existence of global attractors for the Kolmogorov-Spieqel-Sivashinsky equation in $H^3_{per}(\Omega)$ and $H^4_{per}(\Omega)$.

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**Key Words:** Kolmogorov-Spieqel-Sivashinsky equation, strong solution, global attractor

## 1. Introduction

The main purpose of this paper is to consider the existence of global attractors of the semigroup associated with the strong solution for the Kolmogorov-Spieqel-Sivashinsky(KSS equation in short) equation. We are concerned with the following KSS equation:

$$\frac{\partial u}{\partial t} + \alpha\triangle^2 u + \beta\triangle u + \nabla\cdot\overrightarrow{f(u)} + \triangle\varphi(u) + \nu u + h(x) = 0, \quad \text{in } \Omega\times\mathbb{R}^+, \quad (1.1)$$

with the initial condition

$$u(x,0) = u_0(x), \quad \text{in} \quad \Omega \quad (1.2)$$

and the periodic boundary condition

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$$\Omega = \prod_{i=1}^{n}(0, L_i), n \leq 3 \quad \text{and } u \text{ is } \Omega\text{-periodic}, \tag{1.3}$$

where $\alpha, \beta, \nu$ are physical parameters, $\alpha > 0,\ \beta > 0,\ \nu > 0,\ \varphi(u)$ is a given function, $\overrightarrow{f(u)} = \{f_1(u), \ldots, f_n(u)\}$ a vector-valued function.

Equation (1.1) has been derived by many authors such as Depassier and Spieqel [2], Depassier [1], Poyet [6], and the references therein. But as we know, the best result on the existence of the global attractors is obtained by Guo and Wang [4] recently, they gave the existence of the global attractors in $H^2_{per}(\Omega)$.

In the present paper, we improve the work by obtaining the existence of the global attractor of the KSS equation in $H^3_{per}(\Omega)$ and $H^4_{per}(\Omega)$. On the other hand, for the sake of numerical computations, it would be better if one could establish the existence results for global attractors in spaces with higher regularities. Since the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem (1.1)-(1.3), is not uniformly compact, we cannot expect to obtain the existence of the attractor of the semigroup by use the usual existence theorem of global attractors. Fortunately, in [5], new equivalent conditions of the existence for global attractors have been developed. Applying these conditions, we can obtain the global attractors of the semigroup $\{S(t)\}_{t\geq 0}$ in $H^3_{per}(\Omega)$ and $H^4_{per}(\Omega)$.

For notational convenience, we let $\Omega = \prod_{i=1}^{n}(0, L_i),\ n \leq 3$ and denote by $\|\cdot\|$ the norm of $H = L^2_{per}(\Omega)$ with the usual inner product $(\cdot,\cdot)$, $\|\cdot\|_P$ denotes the norm of $L^P(\Omega)$ for $1 \leq P < \infty(\|\cdot\|_2 = \|\cdot\|)$, $\|\cdot\|_Y$ denotes the norm of any Banach space $Y$.

For the problem (1.1)-(1.3), we know from [3] that for $u_0$ given in $H^3_{per}(\Omega)$ and for any $T > 0$, there exists a unique solution $u$ of (1.1)-(1.3) satisfying

$$u \in C([0,T]; H^3_{per}(\Omega)) \cap L^2([0,T]; H^3_{per}(\Omega)),$$

and if $u_0 \in H^4_{per}(\Omega)$, then

$$u \in C([0,T]; H^4_{per}(\Omega)) \cap L^2([0,T]; H^4_{per}(\Omega)).$$

This theorem admits to define the $C^0$ semigroup

$$S(t) : u_0 \to u(t)$$

in $H^3_{per}(\Omega)$ and $H^4_{per}(\Omega)$, which is depend on the initial data $u_0$ in the space.

## 2. Attractor in $H^3_{per}(\Omega)$

First, by summarizing the results in [5], we have

**Lemma 2.1.** (see [5]) *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on a Hilbert space $H$. Assume that $\{S(t)\}_{t\geq 0}$ satisfies the following dissipativity and compactness conditions:*

*(1) $\{S(t)\}_{t\geq 0}$ has a bounded absorbing set $\mathcal{U}$;*

*(2) for any $\varepsilon > 0$ and bounded subset $B$ of $H$, there exist a $t(B) > 0$ and a finite dimensional subspace $H_1$ of $H$ such that $\{\|PS(t)B\|\}_{t\geq 0}$ is bounded, and*

$$\|(I-P)S(t)x\| \leq \varepsilon, \qquad \text{for } t \geq t(B),\ x \in B,$$

*where $P : H \to H_1$ is orthogonal projection.*

Then $\{S(t)\}_{t\geq 0}$ has a global attractor $\mathcal{A}$.

**Lemma 2.2.** (see [4]) *Assume that $u_0 \in H^2_{per}(\Omega), h \in L^2_{per}(\Omega)$, and*

*(1) $4\alpha\nu > \beta^2$;*

*(2) $\varphi' \leq 0,\ |\varphi^{(k)}| \leq C_1|u|^{p+1-k},\ k = 1, 2, \quad 0 \leq p \leq \frac{4}{n}$;*

*(3) $|\overrightarrow{f(u)}| \leq C_2|u|^{q-k},\ k = 0, 1, \quad 1 \leq q \leq 1 + \frac{6}{n},\ n \leq 3$.*

Then the $\omega$-limit set,

$$\mathcal{A} = \omega(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B}$$

is a compact attractor for $\{S(t)\}_{t\geq 0}$ on $H^2_{per}(\Omega)$, where $B = \{u \in H^2_{per}(\Omega) : \|u\| \leq K_1\}$ is an absorbing set of $\{S(t)\}_{t\geq 0}$ in $H^2(\Omega)$.

In the following, we will gave the existence of the global attractor in $H^3_{per}(\Omega)$.

**Lemma 2.3.** *Assume that $u_0 \in H^3_{per}(\Omega)$ and the conditions of Lemma 2.2 hold, then the semigroup $\{S(t)\}_{t\geq 0}$ associated with the problem (1.1)-(1.3) possesses an absorbing ball $B_0(0, \rho_0)$ centered at 0 of radius $\rho_0$ in $H^3_{per}(\Omega)$ which absorbs bounded subsets $B$ of $H^3_{per}(\Omega)$, i.e., there exists a $t_0$, such that $\|u\|_{H^3} \leq \rho_0$ for any $u \in B$ and $t \geq t_0$.*

The proof of Lemma 2.3 is omitted since it is same as in [4]. Now we can begin with proving the semigroup $\{S(t)\}_{t\geq 0}$ has a global attractor in $H^3_{per}(\Omega)$.

Let $\{\omega_k\}$ be an orthogonal basis of $L^2_{per}(\Omega)$ which consists of eigenvectors of $\Delta^2$. The corresponding eigenvalues are denoted by $\lambda_k,\ k = 1, 2, \cdots$. Then $\{\omega_k\}$ is also an orthogonal basis of $H^3_{per}(\Omega)$. We write

$$H_m = \text{span}\{\omega_1, \cdots, \omega_m\},$$

$P_m : H \to H_m$ be the projection and $Q = I - P_m$. then for any $u$

$$u = (u_1, u_2) = (P_m, Qu).$$

Taking the inner product of (1.1) with $-\Delta^3 u_2$ in $H$. Similar to the proof of

Lemma 2.3, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\triangle u_2\|^2+\alpha\|\nabla\triangle^2 u_2\|^2+2\nu\|\nabla\triangle u_2\|^2\le C_1\|\nabla\triangle u_2\|^2+C_2\,, \tag{2.1}$$

it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\triangle u_2\|^2+\alpha\lambda_{m+1}^2\|\nabla\triangle u_2\|^2\le C_1\rho_0^2+C_2\,. \tag{2.2}$$

By Gronwall Lemma, we have

$$\begin{aligned}\|\nabla\triangle u_2(t)\|^2\le{}&\|\nabla\triangle u_2(t_0)\|^2\exp(-\alpha\lambda_{m+1}^2(t-t_0))\\&+\frac{C_1\rho_0^2+C_2}{\alpha\lambda_{m+1}^2}(1-\exp(-\alpha\lambda_{m+1}^2(t-t_0)))<\varepsilon^2\,,\end{aligned} \tag{2.3}$$

providing that

$$\lambda_{m+1}>(\frac{2(C_1\rho_0^2+C_2)}{\alpha\varepsilon^2})^{\frac{1}{2}},\qquad t>t_0+\frac{2}{\alpha\lambda_{m+1}^2}\ln(\frac{\sqrt{2}\rho_0}{\varepsilon})\,.$$

So we have the main result of this section with

**Theorem 2.1.** *Assume that the conditions of Lemma 2.3 hold. Then the semigroup $\{S(t)\}_{t\ge0}$ associated with problem (1.1)-(1.3) possesses a global attractor in $H_{per}^3(\Omega)$, which attracts all bounded subsets of $H_{per}^3(\Omega)$ in the norm of $H_{per}^3(\Omega)$.*

## 3. Attractor in $H_{per}^4(\Omega)$

**Theorem 3.1.** *Assume that $u_0\in H_{per}^4(\Omega)$ and the conditions of Lemma 2.2 hold, Then the semigroup $\{S(t)\}_{t\ge0}$ associated with problem (1.1)-(1.3) possesses an absorbing ball $B_1(0,\rho_1)$ centered at 0 of radius $\rho_1$ in $H_{per}^4(\Omega)$ which absorbs any bounded subsets of $H_{per}^4(\Omega)$.*

*Proof.* Taking the inner product of (1.1) with $\triangle^4 u$ in $H$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle^2 u\|^2+\alpha\|\triangle^3 u\|^2+\nu\|\triangle^2 u\|^2={}&\beta\|\nabla\triangle^2 u\|^2-(\nabla\cdot\overrightarrow{f(u)},\triangle^4 u)\\&-(\triangle\varphi(u),\triangle^4 u)-(h,\triangle^4 u)\,.\end{aligned} \tag{3.1}$$

Since

$$\beta\|\nabla\triangle^2 u\|^2=-\beta(\triangle^2 u,\triangle^3 u)\le\frac{\alpha}{8}\|\triangle^3 u\|^2+\frac{2\beta}{\alpha}\|\triangle^2 u\|^2\,, \tag{3.2}$$

$$\begin{aligned}-(\nabla\cdot\overrightarrow{f(u)},\triangle^4 u)&=-(\triangle(\nabla\cdot\overrightarrow{f(u)}),\triangle^3 u)\\&\le\|f'''(u)\|_\infty\|\nabla u\|_6^3\|\triangle^3 u\|+3\|f''(u)\|_\infty\|\nabla u\|_4\|\triangle u\|_4\|\triangle^3 u\|\end{aligned}$$

$$+\,\|f'(u)\|_\infty\|\nabla\triangle u\|\|\triangle^3 u\|\,. \tag{3.3}$$

By the Sobolev interpolation inequality

$$\begin{aligned}
\|\nabla u\|_6 &\le C\|\nabla u\|^{1-\frac{n}{15}}\|\triangle^3 u\|^{\frac{n}{15}}\,,\\
\|\nabla u\|_4 &\le C\|\nabla u\|^{1-\frac{n}{20}}\|\triangle^3 u\|^{\frac{n}{20}}\,,\\
\|\triangle u\|_4 &\le C\|\nabla u\|^{\frac{4}{5}-\frac{n}{20}}\|\triangle^3 u\|^{\frac{1}{5}+\frac{n}{20}}\,,\\
\|\nabla\triangle u\| &\le C\|\nabla u\|^{\frac{3}{5}}\|\triangle^3 u\|^{\frac{2}{5}}\,,
\end{aligned} \tag{3.4}$$

we see that

$$\begin{aligned}
-(\nabla\cdot\overrightarrow{f(u)},\triangle^4 u) &\le C\|\nabla u\|^{3-\frac{n}{5}}\|\triangle^3 u\|^{1+\frac{n}{5}} + 3C\|\nabla u\|^{\frac{9}{5}-\frac{n}{10}}\|\triangle^3 u\|^{\frac{6}{5}+\frac{n}{10}}\\
&\quad + C\|\nabla u\|^{\frac{3}{5}}\|\triangle^3 u\|^{\frac{7}{5}} \le \frac{\alpha}{8}\|\triangle^3 u\|^{\frac{7}{5}}\,,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
-(\triangle\varphi(u),\triangle^4 u) &= -(\triangle^2\varphi(u),\triangle^3 u)\\
&\le (\|\varphi''''(u)\|_\infty\|\nabla u\|_8^4 + 6\|\varphi'''(u)\|_\infty\|\nabla u\|_8^2\|\triangle u\|_4 + 3\|\varphi''(u)\|_\infty\|\triangle u\|_4^2\\
&\quad + 4\|\varphi''(u)\|_\infty\|\nabla u\|_4\|\nabla\triangle u\|_4 + \|\varphi'(u)\|_\infty\|\triangle^2 u\|)\|\triangle^3 u\|\,.
\end{aligned} \tag{3.6}$$

By (3.4) and

$$\begin{aligned}
\|\nabla u\|_8 &\le C\|\nabla u\|^{1-\frac{3}{40}n}\|\triangle^3 u\|^{\frac{3}{40}n}\,,\\
\|\nabla\triangle u\| &\le C\|\nabla u\|^{\frac{3}{5}-\frac{n}{20}}\|\triangle^3 u\|^{\frac{2}{5}+\frac{n}{20}}\,.
\end{aligned} \tag{3.7}$$

We get

$$\begin{aligned}
(-\triangle\varphi(u),\triangle^4 u) &\le C\|\nabla u\|^{\frac{37}{10}n}\|\triangle^3 u\|^{1+\frac{3}{10}n}\\
&\quad + 6C\|\nabla u\|^{\frac{14}{5}-\frac{n}{5}}\|\triangle^3 u\|^{\frac{65+\frac{n}{5}}{+}} 3C\|\nabla u\|^{\frac{8}{5}-\frac{n}{10}}\|\triangle^3 u\|^{\frac{7}{5}+\frac{n}{10}}\\
&\quad + 4c\|\nabla u\|^{\frac{8}{5}-\frac{n}{10}}\|\triangle^3 u\|^{\frac{3}{5}+\frac{n}{10}} + C\|\triangle^2 u\|\|\triangle^3 u\| \le \frac{\alpha}{8}\|\triangle^3 u\|^2 + C_2\|\triangle^2 u\|^2\\
&\quad + C_3\,,
\end{aligned} \tag{3.8}$$

$$-(h(x),\triangle^4 u) = -(\triangle h(x),\triangle^3 u) \le \|h\|_{H^2}\|\triangle^3 u\| \le \frac{\alpha}{8}\|\triangle^3 u\|^2 + C_4\,. \tag{3.9}$$

By (3.1), (3.2), (3.5), (3.8), (3.9), we see that

$$\frac{d}{dt}\|\triangle^2 u\|^2 + \alpha\|\triangle^3 u\|^2 + 2\nu\|\triangle^2 u\|^2 \le C_5\|\triangle^2 u\|^2 + C_6\,, \tag{3.10}$$

$$\frac{d}{dt}\|\triangle^2 u\|^2 \le C_5\|\triangle^2 u\|^2 + C_6\,.$$

From [4], we know that

$$\frac{d}{dt}\|\triangle u\|^2 + \alpha\|\triangle^2 u\|^2 + 2\nu\|\triangle u\|^2 \le C\,,\quad \forall t \ge t_*\,. \tag{3.11}$$

For $r > 0$ fixed, we integrate this relation between $t$ and $t+r$ and obtain

$$\int_t^{t+r} \|\triangle^2 u(s)\|^2 ds \le \frac{1}{\alpha}\|\triangle u(t)\|^2 + Cr \le C_7, \quad \forall t \ge t_* . \tag{3.12}$$

By uniform Gronwall Lemma, we obtain

$$\begin{aligned} \|\triangle^2 u(t)\|^2 &\le \rho_1^2, \quad \forall t \ge \max\{t_* + r\} \triangleq t_1 \\ &\triangleq (\frac{C_7}{r} + C_6)\exp(C_5) . \end{aligned} \tag{3.13}$$

The proof is completed. □

**Theorem 3.2.** *Assume that the conditions of Theorem 3.1 hold, then the semigroup $\{S(t)\}_{t\ge 0}$ associated with problem (1.1)-(1.3) possesses a global attractor in $H^4_{per}(\Omega)$, which attracts all bounded subsets of $H^4_{per}(\Omega)$ in the norm of $H^4_{per}(\Omega)$.*

*Proof.* Let $\lambda_k, \omega_k,\ k = 1, 2, \cdots$ be as in the proof of Theorem 2.1. Then $\{\omega_k\}$ is also an orthogonal basis of $H^4_{per}\Omega$.

Take the inner product in $H$ of (1.1) with $\triangle^4 u_2$, after a computation which is similar to the proof of Theorem 3.1, we obtain

$$\frac{d}{dt}\|\triangle^2 u_2\|^2 + \alpha\|\triangle^3 u_2\|^2 + 2\nu\|\triangle^2 u_2\|^2 \le C_5\|\triangle^2 u_2\|^2 + C_6 . \tag{3.14}$$

It follows that

$$\frac{d}{dt}\|\triangle^2 u_2\|^2 + \alpha\lambda_{m+1}^2\|\triangle^2 u_2\|^2 \le C_5\rho_1^2 + C_6 . \tag{3.15}$$

By Gronwall Lemma, we have

$$\begin{aligned} \|\triangle^2 u_2(t)\|^2 \le{}& \|\triangle^2 u_2(t_1)\|^2 \exp(-\alpha\lambda_{m+1}^2(t-t_1)) \\ &+ \frac{C_5\rho_1^2 + C_6}{\alpha\lambda_{m+1}^2}(1 - \exp(-\alpha\lambda_{m+1}^2(t-t_1))) < \varepsilon^2 , \end{aligned} \tag{3.16}$$

provided that

$$\lambda_{m+1} > (\frac{2(C_5\rho_1^2 + C_6)}{\alpha\varepsilon^2})^{\frac{1}{2}}, \quad t > t_1 + \frac{2}{\alpha\lambda_{m+1}^2}\ln(\frac{\sqrt{2}\rho_1}{\varepsilon}) .$$

The proof is complete. □

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