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ON THIRD-ORDER EVOLUTION BOUNDARY-VALUE PROBLEMS

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Abstract: We consider linear evolution boundary-value problems for third-order systems in a half-space of $\Re^d$, with Neumann-type homogeneous boundary conditions, both with constant and variable coefficients. As far as the constant coefficient case is concerned, by performing a Fourier-Laplace transform, we discuss the well-posedness of the problem in the Sobolev space H^2, proving a necessary condition (Lopatinskii condition) and a sufficient condition. This last statement is established through the application of Hille-Yosida Theorem. In the case where the system has variable coefficients, we focus on the well-posedness of the boundary-value problem in a half-space, by means of Hille-Yosida result again.

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1. Introduction

The present paper is devoted to the study of linear evolution boundary-value problems in a half-space, for third-order systems. Let us denote by Ω the half-space $\Omega = \Re^{d-1} \times (0, \infty)$; and by u the vector field $u : \Omega \longrightarrow \Re^d$. Let the real function W be the sum of the following forms $W(u, \nabla_x u, D^2 u) = W_1(D^2 u) + W_2(\nabla_x u) + W_3(u, \nabla_x u) + W_4(u)$, where W_2 is a quadratic form on the space $M_{d\times d}(\Re)$ of $d \times d$ matrices with real entries, W_3 is a bilinear form defined on $\Re^d \times M_{d\times d}(\Re)$, W_4 is a quadratic form on $\Re^d$, W_1 is a quadratic form on the space $M_{d\times d\times d}(\Re)$ of $d \times d \times d$ tensors.

Let Q be the third-order differential operator $Q : H^2(\Omega)^d \longrightarrow H^{-1}(\Omega)^d$, defined by

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$$(Q[u])_j = -\sum_{\alpha=1}^{d} \partial_\alpha \left(\sum_{\beta=1}^{d} \frac{\partial W_1}{\partial F_{\alpha\beta j}} + \frac{\partial W_2}{\partial F_{\alpha j}} + \frac{\partial W_3}{\partial F_{\alpha j}} \right) + \frac{\partial W_4}{\partial F_j}, \qquad j = 1, \cdots, d; \quad (1)$$

and B the Neumann-type boundary operator $B : H^2(\Omega)^d \longrightarrow H^{-1/2}\ (\partial\Omega)^d$, defined by

$$(B[u])_j = \sum_{\beta=1}^{d} \frac{\partial W_1}{\partial F_{d\beta j}} + \frac{\partial W_2}{\partial F_{dj}} + \frac{\partial W_3}{\partial F_{dj}}, \ \ j = 1, \cdots, d. \quad (2)$$

We shall study the evolution boundary-value problem with homogeneous boundary condition

$$\begin{cases} \partial_t{}^2 u + Q[u] = f, & x \in \Omega,\ t \in \Re, \\ B[u] = 0, & x \in \partial\Omega,\ t \in \Re; \end{cases} \quad (3)$$

where $f = f(x,t)$ is a given function. If we denote by U the vector field $U = (\partial_t u, \nabla_x u, D^2 u, u)^T$ and by X the linear differential operator $X = (Q[\cdot], -\partial_t \nabla_x, -\partial_t D^2, -\partial_t)^T$, then the problem (1) is equivalent to the following abstract evolution problem

$$\frac{dU}{dt} + X(U) = F, \quad (4)$$

where $F = (f, 0, 0, 0)^T$. If the homogeneous initial value problem (4) is well-posed in the linear space $D(X) = \{u \in H^2(\Omega)^d : Q[u] \in L^2(\Omega)^d;\ B[u] = 0$ on $\partial\Omega\}$, then the operator X generates a continuous semi-group of contractions and the non-homogeneous initial value problem (4) can be solved as usual by means of Duhamel's formula.

If problem (4) is well-posed and the operator X turns out to be monotone on $D(X)$, then, as proved in [2] for second-order initial boundary-value problems, the solution U satisfies the energy estimate

$$e^{-2\gamma T} \|U(T)\|^2_{L^2(\Omega)} + \frac{3}{2}\gamma \int_0^T e^{-2\gamma t} \|U(t)\|^2_{L^2(\Omega)} dt \leq \|U(0)\|^2_{L^2(\Omega)} + \frac{2}{\gamma} \int_0^T e^{-2\gamma t} \|f(t)\|^2_{L^2(\Omega)} dt. \quad (5)$$

We extend to the third-order system (1) the definition of strong well-posedness given for second-order initial boundary-value problems in [4].

Definition 1.1. The boundary-value problem (1) is said to be strongly well-posed if it satisfies the estimate (5).

In the framework of the problems studied in [2] and [3], we shall prove, after performing a Fourier-Laplace transform, a necessary condition (Lopatinskii condition) for the strong well-posedness of (1). Furthermore, we shall write the transformed system in equivalent form as a first-order linear system and by applying Hille-Yosida Theorem, we shall establish a sufficient condition for the well-posedness of the homogeneous boundary-value problem (1). The last section of the paper is devoted to the study of the third-order variable-coefficients evolution boundary-value problem

$$\begin{cases} \partial_t{}^2 u + Q[x,u] = f, & x \in \Omega,\ t \in \Re, \\ B[x,u] = 0, & x \in \partial\Omega,\ t \in \Re; \end{cases} \tag{6}$$

here the associated form W is given by $W(x,u,\nabla_x u, D^2u) = W_1(x,D^2u) + W_2(x,\nabla_x u) + W_3(x,u,\nabla_x u) + W_4(x,u)$ the operator $Q : \Omega \times H^2(\Omega)^d \longrightarrow H^{-1}(\Omega)^d$, is defined by

$$(Q[x,u])_j = -\sum_{\alpha=1}^{d} \partial_\alpha \left(\sum_{\beta=1}^{d} \frac{\partial W_1}{\partial F_{\alpha\beta j}} + \frac{\partial W_2}{\partial F_{\alpha j}} + \frac{\partial W_3}{\partial F_{\alpha j}} \right) + \frac{\partial W_4}{\partial F_j}, \quad j = 1, \cdots, d; \tag{7}$$

and the boundary operator $B : \Omega \times H^2(\Omega)^d \longrightarrow H^{-1/2}(\partial\Omega)^d$, is defined by

$$(B[x,u])_j = \sum_{\beta=1}^{d} \frac{\partial W_1}{\partial F_{d\beta j}} + \frac{\partial W_2}{\partial F_{dj}} + \frac{\partial W_3}{\partial F_{dj}}, \ \ j = 1, \cdots, d. \tag{8}$$

We assume that the functions $W_1(x,\cdot), W_2(x,\cdot), W_3(x,\cdot,\cdot), W_4(x,\cdot)$, for every fixed $x \in \Omega$, enjoy the same properties as in the constant coefficients problem (1). Similarly to the study carried out in [3], the well-posedness of the boundary-value problem (6) relies on the well-posedness of BVPs with frozen coefficients.

Thanks to Hille-Yosida Theorem, we shall establish the existence of a unique solution of the IBVP (6) in the Hilbert space

$$\left\{ u \in H^2(\Omega)^d : Q[x,u] \in L^2(\Omega)^d;\ B[x,u] = 0 \text{ on } \partial\Omega \right\}.$$

2. Fourier-Laplace Analysis

As explained in the Introduction, we shall study the well-posedness of the constant coefficient IBVP (1) in the half-space Ω, by means of the Fourier-Laplace transform. Let $x \in \Re^d$, $x = (x_1, \cdots, x_d)$ and denote by $y = (x_1, \cdots, x_{d-1})$ the tangential variables. If we perform a Fourier transform in the tangential variables and a Laplace transform in the time-variable, we obtain the following collection of BVPs

$$\begin{cases} \tau^2 v + \hat{Q}_\eta[v] = 0, \\ \hat{B}_\eta[v] = 0; \end{cases} \tag{9}$$

where $\eta \in \Re^{d-1}$, $\tau \in \mathbf{C}$; $v = (\omega, v_d)$ is the unknown function; $\hat{Q}_\eta$ and $\hat{B}_\eta$ are the transformed operator of Q and B, respectively. The operator $\hat{Q}_\eta$ is defined by

$$\hat{Q}_\eta[v] = K_\eta i v + H_\eta v' - L_\eta i v'' - M v''' - \Lambda v'' - i(A_\eta + A_\eta^T)v' + \Sigma_\eta v - iC_\eta v - \Gamma v' + Dv; \tag{10}$$

where $K_\eta, H_\eta, L_\eta, M, \Lambda, A_\eta, A_\eta^T, \Sigma_\eta, C_\eta, \Gamma, D \in M_{d\times d}(\Re)$; K_η is cubic in η, H_η and Σ_η are quadratic in η; L_η, A_η, C_η are linear in η; the transformed boundary operator $\hat{B}_\eta$ takes the following form

$$\hat{B}_\eta[v] = -H_\eta v(0) + L_\eta i v'(0) + M v''(0) + \Lambda v'(0) + i(A_\eta + A_\eta^T)v(0) + \Gamma v(0). \tag{11}$$

As far as the estimate (5) is concerned, it has been proved in [2] that, through a Fourier-Laplace transform, (5) becomes equivalent to the following estimate

$$\begin{aligned} &e^{-2\gamma T}\|\partial_t v, \eta\otimes v, \eta\otimes\eta\otimes v, v(T)\|^2_{L^2(\Omega)} \\ &\qquad + \frac{3}{2}\gamma\int_0^T e^{-2\gamma t}\|\partial_t v, \eta\otimes v, \eta\otimes\eta\otimes v, v(t)\|^2_{L^2(\Omega)}dt \\ &\le \|\partial_t v, \eta\otimes v, \eta\otimes\eta\otimes v, v(0)\|^2_{L^2(\Omega)} + \frac{2}{\gamma}\int_0^T e^{-2\gamma t}\|f_\eta(t)\|^2_{L^2(\Omega)}dt; \end{aligned} \tag{12}$$

where $f_\eta = \partial_t^2 v + \hat{Q}_\eta[v]$. The estimate (12) turns out to be equivalent to the same estimate with norms in $L^2(\Re^+)$ instead of $L^2(\Omega)$ as proved in [2]. We discuss now the well-posedness of the problem (9) by means of the techniques introduced in [4], [5], [2] for second-order initial boundary-value problems. Similarly as for the systems studied in [2], we establish the following necessary condition for the well-posedness (Lopatinskii condition).

Proposition 2.1. *Assume that the boundary-value problem (9) is strongly well-posed. Then for every pair (τ, η) with $\eta \in \mathfrak{R}^{d-1}$, $\tau \in \mathbf{C}$, in the cases where either $Re\tau > 0$ or $\tau = 0$ and $\eta \neq 0$, the null function is the only solution of (9), which vanishes at $+\infty$.*

The result can be proved in the same way as the Lopatinskii condition for second-order BVPs (see reference [2]). A sufficient condition for the well-posedness of (1) will be proved below after rewriting system (9) in equivalent form as a first-order linear system. Let us denote by Y the vector field $Y = (v, v', v'')^T$, $Y \in \mathbf{C}^{3d}$. We state the following result.

Proposition 2.2. *Consider system (9) and assume the following conditions are satisfied:*

(i) the matrix M is non-singular;

(ii) the matrices $K_\eta, H_\eta, L_\eta, M, C_\eta, \Gamma$ are symmetric;

(iii) for every $\rho \in \mathfrak{R}$ and $\eta \in \mathfrak{R}^{d-1}$, the matrix $P(\eta, \rho) = \rho^2\Lambda - \rho(A_\eta + A_\eta^T) + \Sigma_\eta + D$ is positive definite.

Then:

1) the system (9) is equivalent to the first-order linear system $Y' = N(\tau, \eta)Y$, where $N(\tau, \eta)$ is a suitable matrix of $M_{3d\times 3d}(\mathbf{C})$;

2) the solution space of (9) splits as the direct sum of the stable space $E_-(\tau, \eta)$ and the unstable space $E_+(\tau, \eta)$, related to the matrix $N(\tau, \eta)$, provided that $-\tau^2$ is not an eigenvalue of $P(\eta, rho)$.

Proof. If the matrix M is non-singular, then the system (9) can be written in the form

$$v''' = M^{-1}(\tau^2 v + K_\eta iv + H_\eta v' - L_\eta iv'' - Mv''' - \Lambda v'' - i(A_\eta + A_\eta^T)v' \\ + \Sigma_\eta v - iC_\eta v - \Gamma v' + Dv).$$

Thus there exists a matrix $N(\tau, \eta) \in M_{3d\times 3d}(\mathbf{C})$ such that the system is equivalent to the first-order linear system $Y' = N(\tau, \eta)Y$. As a result, the space of the solutions of (9) has dimension $3d$. We prove now that the matrix $N(\tau, \eta)$ does not admit pure imaginary eigenvalues. Let us suppose that the matrix $N(\tau, \eta)$ admits the eigenvalue $i\rho$, for some $\rho \in \mathfrak{R}$. Hence there exists $v_0 \in \mathbf{C}^d$ such that the function $v(x_d) = e^{i\rho x_d}v_0$, provides a solution to the system (9). Substituting in the equations (9) and multiplying by v_0^T, we obtain

$$v_0^T(\tau^2 I_d + \rho^2\Lambda + \rho(A_\eta + A_\eta^T) + \Sigma_\eta + D)v_0 \\ + iv_0^T(K_\eta + \rho H_\eta + \rho^2 L_\eta + \rho^3 M - C_\eta - \rho\Gamma)v_0 = 0.$$

Thanks to (ii), the vector $v_0^T(K_\eta + \rho H_\eta + \rho^2 L_\eta + \rho^3 M - C_\eta - \rho\Gamma)v_0$ is real; thus, $v_0^T(\tau^2 I_d + \rho^2\Lambda + \rho(A_\eta + A_\eta^T) + \Sigma_\eta + D)v_0 = 0$. For $-\tau^2$ is not an eigenvalue of $P(\eta,\rho)$, the matrix $P(\eta,\rho) + \tau^2 I_d$ is positive definite and v_0 turns out to be the null vector; whence the space of the solutions of (9) is equivalent to the direct sum of the stable and unstable spaces of $N(\tau,\eta)$. □

Similarly to second-order IBVPs discussed in [2], we shall establish a sufficient condition for the well-posedness of (9) by proving that the operator X, defined in Introduction, is a maximal monotone operator on $D(X)$. The application of Hille-Yosida Theorem will ensure the existence of the solution of the evolution boundary-value problem (1).

Theorem 2.1. *Let $D(X)$ be the functional space $D(X) = \{u \in H^2(\Omega)^d : Q[u] \in L^2(\Omega)^d;\, B[u] = 0\, on\, \partial\Omega\}$ and consider the IBVP (1) under the assumptions of Proposition 2.2. In addition, assume the following conditions are satisfied:*

(i) for every $u \in D(X)$,

$$\int_\Omega \left[W_2(\nabla_x u) + W_3(u, \nabla_x u) + W_4(u) + \left\langle \sum_{\alpha,\beta=1}^{d} \frac{\partial W_1(D^2 u)}{\partial F_{\alpha\beta}}, \partial_\alpha u \right\rangle \right] dx \geq 0; \quad (13)$$

(ii) there exists a positive real constant γ_η such that for every $v \in E_-(1,\eta)$

$$\int_0^{+\infty} (\langle K_\eta i v - B_\eta i v + D v + \Sigma_\eta v + v, v\rangle + \langle -H_\eta v + L_\eta i v' + M v'' + \Lambda v' + i(A_\eta + A_\eta^T)v + \Gamma v, v'\rangle) dx_d \geq \gamma_\eta \|v\|^2_{H^2(\mathfrak{R}^+)}; \quad (14)$$

(iii) if $p_1 : \mathbf{C}^{3d} \longrightarrow \mathbf{C}^d$ is the projection operator, then $\{p_1(E_-(1,\eta))\}^\perp \bigcap \left\{\hat{B}_\eta[v] : v \in H^2(\mathfrak{R}^+)^d\right\} = \emptyset$, for every $\eta \in \mathfrak{R}^{d-1}$.

Then the homogeneous boundary value problem (1) turns out to be well-posed in $D(X)$.

Proof. We prove first that X is a monotone operator on $D(X)$. We consider the system (9) and multiply by v. Integrating we obtain

$$\int_0^{+\infty} Re \left\langle \hat{Q}_\eta[v], v \right\rangle dx_d = \int_0^{+\infty} Re\, (\langle K_\eta i v + H_\eta v' - L_\eta i v'' - M v''' - \Lambda v'' - i(A_\eta + A_\eta^T)v' - \Gamma v' - C_\eta i v + D v + \Sigma_\eta v, v\rangle\, dx_d = Re \int_0^{+\infty} \langle K_\eta i v - C_\eta i v$$

$$+Dv+\Sigma_\eta v, v\rangle\, dx_d + Re\int_0^{+\infty}\left\langle -H_\eta v + L_\eta i v' + M v'' + \Lambda v' + i(A_\eta + A_\eta^T)v \right.$$
$$\left. +\Gamma v, v'\right\rangle dx_d. \quad (15)$$

Because of Plancherel's Theorem and condition (i), we obtain

$$\int_{\Re^{d-1}}\int_0^{+\infty} Re(\langle K_\eta i v - C_\eta i v + Dv + \Sigma_\eta v, v\rangle + \langle -H_\eta v + L_\eta i v' + M v''$$
$$+\Lambda v' + i(A_\eta + A_\eta^T)v + \Gamma v, v'\rangle)dx = \int_\Omega [W_2(\nabla_x u) + W_3(u, \nabla_x u) + W_4(u)$$
$$+\left\langle \sum_{\alpha,\beta=1}^{d} \frac{\partial W_1(D^2 u)}{\partial F_{\alpha\beta}}, \partial_\alpha u\right\rangle\Big]\, dx \geq 0\,, \quad (16)$$

whence the operator X turns out to be monotone on $D(X)$.

Due to the result of Proposition 2.2, the system (9) is equivalent to the first-order linear system $Y' = N(\tau, \eta)Y$, as $\tau = 1$.

Let $Y_0 = (v_0, v_0', v_0'')^T$ be a fixed vector in $E_-(1, \eta)$. In that Y_0 decays to zero as $x \to +\infty$, we obtain by means of similar computations as carried out above,

$$\int_0^{+\infty}\left\langle \hat{Q}_\eta[v_0] + v_0, v_0\right\rangle dx_d$$
$$= \int_0^{+\infty}\langle K_\eta i v_0 - C_\eta i v_0 + D v_0 + \Sigma_\eta v_0 + v_0, v_0\rangle\, dx_d$$
$$+\int_0^{+\infty}\langle -H_\eta v_0 + L_\eta i v_0' + M v_0'' + \Lambda v_0' + i(A_\eta + A_\eta^T)v_0 + \Gamma v_0, v_0'\rangle\, dx_d$$
$$-\left\langle \hat{B}_\eta[v_0], v_0(0)\right\rangle = 0; \quad (17)$$

hence,

$$\int_0^{+\infty}\langle K_\eta i v_0 - C_\eta i v_0 + D v_0 + \Sigma_\eta v_0 + v_0, v_0\rangle\, dx_d$$
$$+\int_0^{+\infty}\langle -H_\eta v_0 + L_\eta i v_0' + M v_0'' + \Lambda v_0' + i(A_\eta + A_\eta^T)v_0 + \Gamma v_0, v_0'\rangle\, dx_d$$
$$= \left\langle \hat{B}_\eta[v_0], v_0(0)\right\rangle. \quad (18)$$

Thanks to the assumption (ii), the sesquilinear form $\left\langle \hat{B}_\eta[v], w\right\rangle$ turns out to be coercive on the space $E_-(1, \eta)$. Let G be a fixed vector valued function

$G = (g, 0_d, 0_d)^T$, $G : \Omega \longrightarrow \mathbf{C}^{3d}$, such that $g \in L^2(\Omega)^d$. For the matrix $N(1,\eta)$ does not have pure imaginary eigenvalues, $\mathbf{C}^{3d} = E_-(1,\eta) \oplus E_+(1,\eta)$. Let us denote by G_-, G_+ the components of G, which belong to the invariant spaces $E_-(1,\eta)$, $E_+(1,\eta)$, respectively. We look for a solution $Y \in D(X)$, to the first-order linear system $Y' = N(1,\eta)Y + G$, in order to prove that X is a maximal operator. Let us denote by $(S(z))_{z\in\mathfrak{R}}$ the semi-group generated by $N(1,\eta)$, $S(z) = \exp(zN(1,\eta))$; we shall prove that the function Y having components Y_-, Y_+, defined by

$$\begin{aligned} Y_-(\eta, x_d) &= S(x_d)Y_0 + \int_0^{x_d} S(x_d - z)G_-(\eta, z)dz, \\ Y_+(\eta, x_d) &= -\int_{x_d}^{+\infty} S(x_d - z)G_+(\eta, z)dz \,, \end{aligned} \tag{19}$$

provides a solution to the system $Y' = N(1,\eta)Y + G$, where $Y_0 = (v_0, w_0, z_0)^T$.

In that $g \in L^2(\Omega)^d$, the functions Y_-, Y_+ turn out to belong to $L^2(\mathfrak{R}^+)^{3d}$, provided that Y_0 is chosen in the stable space $E_-(1,\eta)$. If we can find Y_0 in order to satisfy the homogeneous boundary condition, then the operator X is maximal on $D(X)$. Let Z be a function of $E_-(1,\eta)$ and $v = p_1(Y)$. Hence, $\left\langle \hat{B}_\eta[v], p_1(Z)(0)\right\rangle = \left\langle \hat{B}_\eta[v_0], p_1(Z)(0)\right\rangle + \left\langle \hat{B}_\eta[v^+], p_1(Z)(0)\right\rangle$.

The sesquilinear form $\left\langle \hat{B}_\eta[v_0], p_1(Z)(0)\right\rangle$ is coercive and bounded on the space $E_-(1,\eta)$. Let $F : E_-(1,\eta) \longrightarrow \mathbf{C}$, be the functional defined by $F(z) = -\left\langle \hat{B}_\eta[v^+], p_1(Z)(0)\right\rangle$. Due to Lax-Milgram Theorem, there exists a unique function $\bar{Y}_0 \in E_-(1,\eta)$, such that $F(z) = \left\langle \hat{B}_\eta[\bar{v}_0], p_1(Z)(0)\right\rangle$, for every $Z \in E_-(1,\eta)$. Thus, $\left\langle \hat{B}_\eta[\bar{v}], p_1(Z)(0)\right\rangle = \left\langle \hat{B}_\eta[\bar{v}_0], p_1(Z)(0)\right\rangle + \left\langle \hat{B}_\eta[v^+], p_1(Z)(0)\right\rangle = 0$, for all $Z \in E_-(1,\eta)$. Finally, because of the assumption (iii), we deduce that $\hat{B}_\eta[\bar{v}] = 0$. Since the operator X is monotone and maximal, by applying Hille-Yosida Theorem, the homogeneous problem (1) turns out to be well-posed in $D(X)$. □

Remark 2.1. The assumptions (i) and (ii) in Theorem 2.1 may be substituted by the stronger condition:

(j) there exists a positive constant γ such that for every $u \in D(X)$,

$$\int_\Omega \left[W_2(\nabla_x u) + W_3(u, \nabla_x u) + W_4(u) + \left\langle \sum_{\alpha,\beta=1}^{d} \frac{\partial W_1(D^2u)}{\partial F_{\alpha\beta}}, \partial_\alpha u \right\rangle \right] dx \geq \gamma \|u\|^2_{H^2(\Omega)^d}. \tag{20}$$

3. Boundary-Value Problems for Systems with Variable Coefficients

This section is devoted to the study of the variable-coefficient third-order boundary-value problem (6). Similarly to the sufficient condition for the well-posedness of second-order IBVPs (see [3]), we establish the following result

Theorem 3.1. *Consider the boundary-value problem (6) and denote by* K *the linear space* $K = \left\{u \in H^2(\Omega)^d : Q[x,u] \in L^2(\Omega)^d;\, B[x,u] = 0 \,\text{on}\, \partial\Omega\right\}$. *Assume the following conditions are fulfilled:*

(i) There exist $x_1 \in \Omega$ *and a positive constant* γ *such that, for every* $u \in K$,

$$\int_\Omega \Bigg[W_2(x_1, \nabla_x u) + W_3(x_1, u, \nabla_x u) + W_4(x_1, u) + \left\langle \sum_{\alpha,\beta=1}^{d} \frac{\partial W_1(x_1, D^2 u)}{\partial F_{\alpha\beta}}, \partial_\alpha u \right\rangle \Bigg] dx \geq \gamma \|u\|^2_{H^2(\Omega)^d};$$

(ii) The function $x \to W(x,u,w,z)$ *belongs to the space* $C^1(\Omega)$, *for every fixed* $(u,w,z) \in \mathfrak{R}^d \times \mathfrak{R}^{d^2} \times \mathfrak{R}^{d^3}$. *In addition, there exists a real constant* $\lambda > 0$, *with* $\gamma > 2\lambda(d^2 + d^3 + d^4 + d^6)$, *such that the coefficients of the forms* W_1, W_2, W_3, W_4 *are bounded by* λ *in* Ω.

Then the evolution boundary-value problem (6) turns out to be well-posed in K.

Proof. The problem (6) is equivalent to the following evolution problem in abstract form: $\frac{dU}{dt} + A(U) = 0$, where the vector field U is defined by $U = (\partial_t u, \nabla_x u, D^2 u, u)^T$ and the linear differential operator A is given by $A = (Q[x,\cdot],\, -\partial_t \nabla_x,\, -\partial_t D^2,\, -\partial_t)^T$. We establish the well-posedness of (6) by applying Hille-Yosida Theorem. Let us prove that A is monotone. If $u \in K$, we obtain

$$\begin{aligned}
&\int_\Omega (Q[x,u], u) dx \\
&\quad= \int_\Omega \left\langle -\sum_{\alpha=1}^{d} \partial_\alpha \left(\sum_{\beta=1}^{d} \frac{\partial W_1}{\partial F_{\alpha\beta j}} + \frac{\partial W_2}{\partial F_{\alpha j}} + \frac{\partial W_3}{\partial F_{\alpha j}} \right) + \frac{\partial W_4}{\partial F_j}, u \right\rangle dx \\
&\quad= \int_\Omega \sum_{j=1}^{d} \left(\sum_{\alpha=1}^{d} \left(\sum_{\beta=1}^{d} \frac{\partial W_1}{\partial F_{\alpha\beta j}} + \frac{\partial W_2}{\partial F_{\alpha j}} + \frac{\partial W_3}{\partial F_{\alpha j}} \right) \partial_\alpha u_j + \frac{\partial W_4}{\partial F_j} u_j \right) dx \\
&\quad= \int_\Omega (W_2(x, \nabla_x u) + W_3(x, u, \nabla_x u) + W_4(x, u)) dx
\end{aligned}$$

$$+\int_\Omega \leq \sum_{\alpha,\beta=1}^{d} \left\langle \frac{\partial W_1}{\partial F_{\alpha\beta j}}, \partial_\alpha u \right\rangle dx. \quad (21)$$

Because of the assumption (i), we have

$$\begin{aligned}\int_\Omega &[W_2(x,\nabla_x u) + W_3(x,u,\nabla_x u) \\ &+ W_4(x,u) \pm W_2(x_1,\nabla_x u) \pm W_3(x_1,u,\nabla_x u) \pm W_4(x_1,u)]dx \\ &+ \int_\Omega \sum_{\alpha,\beta=1}^{d} \left\langle \frac{\partial W_1(x,D^2u)}{\partial F_{\alpha\beta}} \pm \frac{\partial W_1(x_1,D^2u)}{\partial F_{\alpha\beta}}, \partial_\alpha u \right\rangle dx \\ &\geq \gamma \|u\|^2_{H^2(\Omega)^d} - 2\lambda(d^2+d^3+d^4+d^6)\|u\|^2_{H^2(\Omega)^d}. \quad (22)\end{aligned}$$

Thanks to (ii), the operator X is monotone on the linear space K. Let $u, v \in K$. Hence,

$$\begin{aligned}\int_\Omega (Q[x,u]+u,v)dx = \int_\Omega \sum_{\alpha=1}^{d} \left(\sum_{\beta=1}^{d} \frac{\partial W_1}{\partial F_{\alpha\beta j}} + \frac{\partial W_2}{\partial F_{\alpha j}} + \frac{\partial W_3}{\partial F_{\alpha j}}, \partial_\alpha v + \right) dx \\ + \int_\Omega \left\langle \frac{\partial W_4}{\partial F_j} + u, v \right\rangle dx - \int_{\Re^{d-1}} (B[x,u],v)dy.\end{aligned}$$

Due to the assumptions, the bilinear form $\int_\Omega (Q[x,u]+u,v)dx$ turns out to be continuous and coercive on K. Let g be a fixed function in $L^2(\Omega)^d$. Thanks to Lax-Milgram Theorem, there exists a unique function $\bar{u} \in K$, such that

$$\int_\Omega (Q[x,\bar{u}]+\bar{u},v)dx = \int_\Omega (g,v)dx, \quad (23)$$

for all $v \in K$.

Since $\bar{u} \in H^2(\Omega)^d$, by Meyers-Serrin Theorem, the function $\bar{u}$ can be approximated in H^2-norm by means of a sequence of functions $(\bar{u}_n)_{n\in\mathbf{N}} \subset H^2(\Omega)^d \bigcap C^\infty(\Omega)^d$. For every function $\phi \in C_0^\infty(\Omega)^d$, we obtain, in view of the previous computations,

$$\begin{aligned}&\int_\Omega (Q[x,\bar{u}_n]+\bar{u}_n,\phi)dx = \\ &\int_\Omega \sum_{\alpha=1}^{d} \left(\sum_{\beta=1}^{d} \frac{\partial W_1}{\partial F_{\alpha\beta j}}(x,D^2\bar{u}_n) + \frac{\partial W_2}{\partial F_{\alpha j}}(x,\nabla_x \bar{u}_n) + \frac{\partial W_3}{\partial F_{\alpha j}}(x,\bar{u}_n,\nabla_x \bar{u}_n), \partial_\alpha \phi \right) dx\end{aligned}$$

$$+\int_\Omega \left(\frac{\partial W_4}{\partial F_j}(x, \bar{u}_n) + \bar{u}_n, \phi \right) dx. \quad (24)$$

Hence, $\lim_{n\to+\infty}(Q[x,\bar{u}_n] + \bar{u}_n) = Q[x,\bar{u}] + \bar{u}$, weakly in $L^2(\Omega)^d$. Because of (23) and the uniqueness of the weak limit, we obtain $Q[x,\bar{u}] + \bar{u} = g$, a.e. in Ω. As a consequence of this result, the operator A turns out to be maximal and by Hille-Yosida Theorem, the initial boundary-value problem (6) is well-posed in the Hilbert space K. □

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