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A VARIATIONAL FORMULATION OF THE CONTROLLED PLATE MODEL

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Abstract: In this paper we study the Mindlin-Timoshenko plate model. The model comes from the so-called Kirchoff plate model after a weakening of the Kirchoff hypothesis by removing the assumption that the filaments of the plate remain perpendicular to the deformed middle surface, but keeping the assumption that they remain straight and undergo no strain deformation.

The derivations of the model in the literature are typically not completely rigorous, but Ciarlet has in [1] shown how this model (and many others) are "correct" in the sense that it is the natural first "term" in an expansion of a full, nonlinear model.

The derivation that follows is rather standard, but with the additional purpose of being able to formulate the boundary control problem in a variational setting there are some modifications in comparison to the derivations from mechanical engineering and the works of Lagnese and Lions, [3], [2]. The question of well-posedness and choice of "state-space" is approached by application of variational theory.

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1. Introduction and Notation

The Mindlin-Timoshenko model describes the elastic motion of a homogeneous

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and isotropic thin plate, the motion is assumed to be elastic, in the sense that no permanent deformation of the plate occurs.

We consider the plate in rectangular coordinates $(x, y, z) \in R^3$, although we will use (x_1, x_2, x_3) to represent the same coordinate whenever index notation is more convenient. As usual the summation convention for repeated indicies is used, thus $\epsilon_{kk} = \sum_{k=1}^{3} \epsilon_{kk}$.

When the plate is in equilibrium it occupies the volume denoted $\Omega \times]\frac{-h}{2}, \frac{h}{2}[$, where h is the thickness of the plate and Ω is called the middle surface. We assume Ω is a bounded subset of R^2 with a sufficiently smooth boundary $\Gamma = \partial\Omega$ (what is meant by sufficiently smooth will be made more precise later).

In order to describe the deformation of the plate when it is bended, we use the displacement vector

$$U = \begin{pmatrix} U_1(x, y, z, t) \\ U_2(x, y, z, t) \\ U_3(x, y, z, t) \end{pmatrix} . \tag{1}$$

In the middle surface $z = 0$ and we will denote the displacement vector by

$$u = \begin{pmatrix} u_1(x, y, t) \\ u_2(x, y, t) \\ u_3(x, y, t) \end{pmatrix} = \begin{pmatrix} U_1(x, y, 0, t) \\ U_2(x, y, 0, t) \\ U_3(x, y, 0, t) \end{pmatrix} . \tag{2}$$

The main assumption for the Mindlin-Timoshenko model is now that straight filaments perpendicular to the middle surface in equilibrium remain straight and do not contract nor expand. To describe the rotation of the filament in the x and y direction we use polar coordinates ψ and φ, respectively (see Figure 1)

We will use this assumption to derive a relation between U and u.

Let $e = (\alpha, \beta, \gamma)$ be a unit vector pointing from $u(x, y, t)$ to $U(x, y, z, t) + (0, 0, z)$, i.e.

$$e = \frac{(U + (0, 0, z)) - u}{|(U + (0, 0, z)) - u|}. \tag{3}$$

With this, α, β and γ can now be expressed as,

$$\gamma = \frac{\cos\psi}{\sqrt{\alpha^2 + \gamma^2}}, \qquad \gamma = \frac{\cos\varphi}{\sqrt{\beta^2 + \gamma^2}}, \quad \alpha = \frac{\sin\psi}{\sqrt{\alpha^2 + \gamma^2}}, \qquad \beta = \frac{\sin\varphi}{\sqrt{\beta^2 + \gamma^2}}$$

(see Figure 1).

Now we denote $a = \frac{1}{\sqrt{\alpha^2 + \gamma^2}}$ and rewrite the above as

$$\alpha = a \sin\psi, \quad \beta = a \frac{\cos\psi \sin\varphi}{\cos\varphi}, \quad \gamma = a \cos\psi . \tag{4}$$

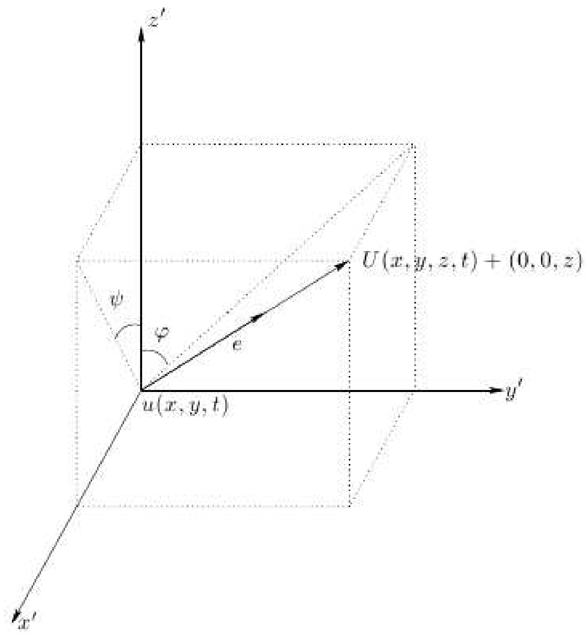

Figure 1: If we think of the filament as being parallel to z' when the plate is in equilibrium, we use ψ and φ to describe the rotation of that filament when the plate is bended, since the filament remains straight and does not expand or contract we get the relative relative position of U with regard to u

Now, inserting into $1 = \alpha^2 + \beta^2 + \gamma^2$ gives us

$$1 = a^2(\sin^2\psi + \frac{\cos^2\psi\sin^2\varphi}{\cos^2\varphi} + \cos^2\psi) \Leftrightarrow a = \left(1 + \frac{\cos^2\psi\sin^2\varphi}{\cos^2\varphi}\right)^{-\frac{1}{2}}. \qquad (5)$$

If we now use the following Taylor expansion for φ and ψ near 0:

$$\frac{\cos^2\psi\sin^2\varphi}{\cos^2\varphi} = \varphi^2 + \frac{2}{3}\varphi^4 - \psi^2\varphi^2 + \frac{1}{3}\psi^4\varphi^2 - \frac{2}{3}\psi^2\varphi^4 + \frac{17}{45}\varphi^6 + \dots$$

we see that a linearization of (5) actually means that $a \approx 1$. If we assume that ψ and φ are small, then $\sin\psi \approx \psi$, $\sin\varphi \approx \varphi$ and $\cos\psi = \cos\varphi \approx 1$, hence (4) becomes

$$\alpha \approx \psi, \qquad \beta \approx \varphi, \qquad \gamma \approx 1\,.$$

Since the filaments do not expand or contract, z is exactly the length $|(U(x,y,z,t)+(0,0,z))-u(x,y,t)|$, thus (3) becomes approximately

$$U = \begin{pmatrix} u_1 + z\psi \\ u_2 + z\varphi \\ w \end{pmatrix}, \tag{6}$$

where $w = u_3$.

The relation between the displacement U and the strain tensor ϵ_{ij} is assumed to be of the form

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right). \tag{7}$$

If we now substitute (6) into the above we get

$$\begin{aligned}
\epsilon_{11} &= \frac{\partial u_1}{\partial x} + z\frac{\partial \psi}{\partial x}\,, \\
\epsilon_{22} &= \frac{\partial u_2}{\partial y} + z\frac{\partial \varphi}{\partial y}\,, \\
\epsilon_{33} &= 0\,, \\
\epsilon_{12} = \epsilon_{21} &= \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + z\left(\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x}\right)\right), \\
\epsilon_{13} = \epsilon_{31} &= \frac{1}{2}\left(\frac{\partial w}{\partial x} + \psi\right), \\
\epsilon_{23} = \epsilon_{32} &= \frac{1}{2}\left(\frac{\partial w}{\partial y} + \varphi\right).
\end{aligned} \tag{8}$$

Let σ_{ij} denote the stress tensor – for homogeneous isotropic plates the stress-strain relations are:

$$\sigma_{ij} = \frac{E}{1+\mu}\left(\epsilon_{ij} + \frac{\mu}{1-2\mu}\epsilon_{kk}\delta_{ij}\right), \tag{9}$$

where E is the Young modulus, μ Poisson's ratio ($0 < \mu < \frac{1}{2}$) and δ_{ij} the Kronecker delta.

It is usual in thin plate theory to neglect the normal stress component σ_{33}, since it is generally small compared to the other stress components, and instead correct $\sigma_{13} = \sigma_{31}$ and $\sigma_{23} = \sigma_{32}$ by a factor $k > 0$ called the shear correction coefficient. By imposing that $\sigma_{33} = 0$ in (9), we find that ϵ_{33} can be expressed in terms of ϵ_{11} and ϵ_{22} in the following way:

$$\epsilon_{33} = -\frac{\mu}{1-\mu}(\epsilon_{11} + \epsilon_{22})\,.$$

Now, substituting this into (9) and using the shear correction coefficient we get a new stress-strain relation

$$\begin{aligned}
\sigma_{11} &= \frac{E}{1-\mu^2}(\epsilon_{11}+\mu\epsilon_{22})\,, \\
\sigma_{22} &= \frac{E}{1-\mu^2}(\mu\epsilon_{11}+\epsilon_{22})\,, \\
\sigma_{33} &= 0\,, \\
\sigma_{12} = \sigma_{21} &= \frac{E}{1+\mu}\epsilon_{12}\,, \\
\sigma_{13} = \sigma_{31} &= k\frac{E}{1+\mu}\epsilon_{13}\,, \\
\sigma_{23} = \sigma_{32} &= k\frac{E}{1+\mu}\epsilon_{23}\,.
\end{aligned} \tag{10}$$

The strain-displacement relations (8) and the above stress-strain relations are what is understood as the Mindlin-Timoshenko plate model.

2. Energy Relations in the M-T Model

As we will use Lagrangian mechanics to deduce the governing equations of the plate we will derive expressions for the mechanical energy in the plate and for the work done on the plate.

2.1. Strain Energy

When the plate is elastically deformed it has potential energy, called the strain energy, defined as

$$V(t) = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{\Omega}\epsilon_{ij}\sigma_{ij}dX\,, \tag{11}$$

where $dX = dxdydz$. Using (10) this can be expressed in terms of the strain components only

$$V(t) = \frac{E}{2(1-\mu^2)}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{\Omega}\epsilon_{11}^2+\epsilon_{22}^2+2\mu\epsilon_{11}\epsilon_{22} + 2(1-\mu)\left(\epsilon_{12}^2+k\epsilon_{13}^2+k\epsilon_{23}^2\right)dX\,.$$

Substituting the strain-displacement relations (8) into the above we get

$$V(t) = \frac{E}{2(1-\mu^2)} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_\Omega \left(\frac{\partial u_1}{\partial x} + z\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} + z\frac{\partial \varphi}{\partial y} \right)^2 + 2\mu \left(\frac{\partial u_1}{\partial x} + z\frac{\partial \psi}{\partial x} \right) \left(\frac{\partial u_2}{\partial y} + z\frac{\partial \varphi}{\partial y} \right) + \frac{1-\mu}{2} \left[\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} + z \left(\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right) \right)^2 + k \left(\frac{\partial w}{\partial x} + \psi \right)^2 + k \left(\frac{\partial w}{\partial y} + \varphi \right)^2 \right] dX.$$

By integrating with respect to z we eliminate all the terms coupling u_1, u_2 and ψ, φ, w, and we arrive at

$$V(t) = \frac{Eh}{2(1-\mu^2)} \int_\Omega \left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + 2\mu \frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial y} + \frac{1-\mu}{2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 + \frac{h^2}{12} \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + 2\mu \frac{\partial \psi}{\partial x}\frac{\partial \varphi}{\partial y} + \frac{1-\mu}{2} \left(\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right)^2 \right] + \frac{1-\mu}{2} k \left[\left(\frac{\partial w}{\partial x} + \psi \right)^2 + \left(\frac{\partial w}{\partial y} + \varphi \right)^2 \right] dxdy. \quad (12)$$

We are now able to split $V(t)$ into a bending component $V_b(t)$, containing the terms involving ψ, φ and w, and a stretching component $V_s(t)$, containing u_1 and u_2.

When deriving the equations of motion for the plate we will only be interested in the bending motion, and from (12) we get

$$V_b(t) = \frac{1}{2} \int_\Omega D \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + 2\mu \frac{\partial \psi}{\partial x}\frac{\partial \varphi}{\partial y} + \frac{1-\mu}{2} \left(\frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial x} \right)^2 \right] + K \left[\left(\frac{\partial w}{\partial x} + \psi \right)^2 + \left(\frac{\partial w}{\partial y} + \varphi \right)^2 \right] dxdy. \quad (13)$$

Here $D = \frac{Eh^3}{12(1-\mu^2)}$ and $K = \frac{kEh}{2(1+\mu)}$ are the *modulus of flexural rigidity* and the *shear modulus*, respectively.

2.2. Kinetic Energy

We can use a procedure similar to the above to uncouple the kinetic energy T into bending and stretching components.

Assuming that the density function of the plate ρ is a constant, the kinetic energy is defined by

$$T(t) = \frac{\rho}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{\Omega} \frac{\partial U_i}{\partial t} \frac{\partial U_i}{\partial t} dX$$

and using (6) we get

$$T(t) = \frac{\rho}{2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{\Omega} \left(\frac{\partial}{\partial t} (u_1 + z\psi) \right)^2 + \left(\frac{\partial}{\partial t} (u_2 + z\varphi) \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 dX.$$

Integrating with respect to z uncouples the above and we get

$$T(t) = T_s(t) + T_b(t)\,,$$

where

$$\begin{aligned} T_s(t) &= \frac{\rho h}{2} \int_{\Omega} \left(\frac{\partial u_1}{\partial t} \right)^2 + \left(\frac{\partial u_2}{\partial t} \right)^2 dxdy\,, \\ T_b(t) &= \frac{\rho h}{2} \int_{\Omega} \frac{h^2}{12} \left(\left(\frac{\partial \psi}{\partial t} \right)^2 + \left(\frac{\partial \varphi}{\partial t} \right)^2 \right) + \left(\frac{\partial w}{\partial t} \right)^2 dxdy \end{aligned} \tag{14}$$

are the stretching and bending components.

2.3. Work

We will assume that there are no external forces on the faces of the plate. The edge $\Gamma \times] -\frac{h}{2}, \frac{h}{2}[$, on the other hand, is subject to a distribution of forces (f_1, f_2, f_3). The work done on the plate is then by definition

$$W(t) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{\Gamma} f_i U_i d\Gamma dz. \tag{15}$$

Using (6) we get

$$W(t) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{\Gamma} f_1(u_1 + z\psi) + f_2(u_2 + z\varphi) + f_3 w d\Gamma dz\,. \tag{16}$$

Now integrating with respect to z enable us to decouple into stretching and bending components as follows

$$W(t) = W_s(t) + W_b(t)\,,$$

$$W_s(t) = \int_\Gamma F_1 u_1 + F_2 u_2 d\Gamma \,,$$
$$W_b(t) = \int_\Gamma m_1 \psi + m_2 \varphi + F_3 w d\Gamma \,, \tag{17}$$

where $F_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i dz$ and $m_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i z dz$.

3. Equations of Motion

The Lagrangian for the plate is defined by

$$L(t) = T(t) - V(t) + W(t) \tag{18}$$

and Hamilton's principle dictates that the motion of the plate is a stationary point of the action

$$\mathcal{L}(u_1, u_2, \psi, \varphi, w) = \int_0^T L(t)dt, \tag{19}$$

that is, the variation satisfies

$$\delta\mathcal{L}(u_1, u_2, \psi, \varphi, w; \tilde{u}_1, \tilde{u}_2, \tilde{\psi}, \tilde{\varphi}, \tilde{w}) = 0 \tag{20}$$

for all admissible directions $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{\psi}$, $\tilde{\varphi}$ and $\tilde{w}$.

Since we were able to uncouple the stretching and bending components of the strain and kinetic energy as well as the work done on the plate, we deduce from (20) that

$$\delta\mathcal{L}_s(u_1, u_2; \tilde{u}_1, \tilde{u}_2) = 0 \,, \tag{21}$$
$$\delta\mathcal{L}_b(\psi, \varphi, w; \tilde{\psi}, \tilde{\varphi}, \tilde{w}) = 0 \,,$$

where

$$\mathcal{L}_s(u_1, u_2; \tilde{u}_1, \tilde{u}_2) = \int_0^T T_s(t) - V_s(t) + W_s(T) dt \,, \tag{22}$$
$$\mathcal{L}_b(\psi, \varphi, w; \tilde{\psi}, \tilde{\varphi}, \tilde{w}) = \int_0^T T_b(t) - V_b(t) + W_b(T) dt \,. \tag{23}$$

From now on we will only consider the bending motion since it is the most relevant from a control point of view.

3.1. The Bending Motion

In order to keep a short notation we will use the following vectors

$$u = (\psi, \varphi, w), \qquad v = (\tilde{\psi}, \tilde{\varphi}, \tilde{w}).$$

The variation of the kinetic energy is

$$\delta \int_0^T T_b(t)dt = \frac{d}{d\epsilon}\frac{1}{2}\int_0^T \int_\Omega D\Bigg[\left(\frac{\partial(\psi+\epsilon\tilde{\psi})}{\partial x}\right)^2 + \left(\frac{\partial(\varphi+\epsilon\tilde{\varphi})}{\partial y}\right)^2 + 2\mu\left(\frac{\partial(\psi+\epsilon\tilde{\psi})}{\partial x}\right)\left(\frac{\partial(\varphi+\epsilon\tilde{\varphi})}{\partial y}\right) + \frac{1-\mu}{2}\left(\frac{\partial(\psi+\epsilon\tilde{\psi})}{\partial y} + \frac{\partial(\varphi+\epsilon\tilde{\varphi})}{\partial x}\right)^2\Bigg] + K\Bigg[\left(\frac{\partial(w+\epsilon\tilde{w})}{\partial x} + (\psi+\epsilon\tilde{\psi})\right)^2 + \left(\frac{\partial(w+\epsilon\tilde{w})}{\partial y} + (\varphi+\epsilon\tilde{\varphi})\right)^2\Bigg]dxdydt\Bigg|_{\epsilon=0} = \int_0^T a(u,v)dt\,, \quad (24)$$

where we have introduced the symmetric bilinear form

$$a(u,v) = a_0(\psi,\varphi,\tilde{\psi},\tilde{\varphi}) + Ka_1(u,v) \qquad (25)$$

with

$$a_0(\psi,\varphi,\tilde{\psi},\tilde{\varphi}) = \int_\Omega D\left[\frac{\partial\psi}{\partial x}\frac{\partial\tilde{\psi}}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\tilde{\varphi}}{\partial y} + \mu\frac{\partial\tilde{\psi}}{\partial x}\frac{\partial\varphi}{\partial y} + \mu\frac{\partial\psi}{\partial x}\frac{\partial\tilde{\varphi}}{\partial y} + \frac{1-\mu}{2}\left(\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial x}\right)\left(\frac{\partial\tilde{\psi}}{\partial y} + \frac{\partial\tilde{\varphi}}{\partial x}\right)\right]dxdy \qquad (26)$$

and

$$a_1(u,v) = \int_\Omega \left(\frac{\partial w}{\partial x} + \psi\right)\left(\frac{\partial\tilde{w}}{\partial x} + \tilde{\psi}\right) + \left(\frac{\partial w}{\partial y} + \varphi\right)\left(\frac{\partial\tilde{w}}{\partial y} + \tilde{\varphi}\right)dxdy. \qquad (27)$$

The variation of the strain energy is

$$\delta \int_0^T V_b = \frac{d}{d\epsilon}\frac{\rho h}{2}\int_0^T \int_\Omega \frac{h^2}{12}\left(\frac{\partial(\psi+\epsilon\tilde{\psi})}{\partial t}\right)^2 + \frac{h^2}{12}\left(\frac{\partial(\varphi+\epsilon\tilde{\varphi})}{\partial t}\right)^2 + \left(\frac{\partial(w+\epsilon\tilde{w})}{\partial t}\right)^2 dxdydt\Bigg|_{\epsilon=0}$$

$$= \int_0^T c(u', v')dt\,, \tag{28}$$

where the symmetric bilinear form

$$c(u,v) = \int_\Omega \rho h \left[\frac{h^2}{12}\psi\tilde{\psi} + \frac{h^2}{12}\varphi\tilde{\varphi} + w\tilde{w}\right] dxdy \tag{29}$$

has been introduced. Calculating the variation of the work we find

$$\begin{aligned}\delta \int_0^T W_b =& \frac{d}{d\epsilon}\int_0^T \int_\Gamma m_1(\psi + \epsilon\tilde{\psi}) + m_2(\varphi + \epsilon\tilde{\varphi}) + F_3(w + \epsilon\tilde{w})d\Gamma dt\bigg|_{\epsilon=0} \\ =& \int_0^T \int_\Gamma m_1\tilde{\psi} + m_2\tilde{\varphi} + F_3\tilde{w}d\Gamma dt. \end{aligned}\tag{30}$$

For convenience we introduce the vector $\kappa = (\kappa_1, \kappa_2, \kappa_3) = (m_1, m_2, F_3)$, and from (21) and (24)-(30) we now have the equation for the motion of the plate:

$$\int_0^T (a(u,v) - c(u',v') + \int_\Gamma \kappa \cdot v d\Gamma)dt = 0. \tag{31}$$

Now we will assume that the forces that influence the plate in order to control it are applied to a subset Γ_0 of the boundary Γ, and we will demand that the solution u is zero on the remaining part Γ_1 of Γ. This means that the plate is clamped along Γ_1. The boundary value problem corresponding to (31) is now derived by an application of Greens formula to $a_0(u,v)$ and $a_1(u,v)$. This yields

$$\begin{aligned} a_0(u,v) = & \int_\Gamma \tilde{\psi} D\left[\left(\frac{\partial\psi}{\partial x} + \mu\frac{\partial\varphi}{\partial y}\right)n_1 + \frac{1-\mu}{2}\left(\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial x}\right)n_2\right] \\ & \tilde{\varphi} D\left[\left(\frac{\partial\varphi}{\partial y} + \mu\frac{\partial\psi}{\partial x}\right)n_2 + \frac{1-\mu}{2}\left(\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial x}\right)n_1\right] d\Gamma \\ & - \int_\Omega \tilde{\psi} D\left[\left(\frac{\partial^2\psi}{\partial x^2} + \mu\frac{\partial^2\varphi}{\partial x\partial y}\right) + \frac{1-\mu}{2}\left(\frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\varphi}{\partial x\partial y}\right)\right] \\ & \tilde{\varphi} D\left[\left(\frac{\partial^2\varphi}{\partial y^2} + \mu\frac{\partial^2\psi}{\partial x\partial y}\right) + \frac{1-\mu}{2}\left(\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\varphi}{\partial x^2}\right)\right] dxdy \end{aligned}\tag{32}$$

and

$$a_1(u,v) = \int_\Gamma \tilde{w}\left[\left(\frac{\partial w}{\partial x} + \psi\right)n_1 + \left(\frac{\partial w}{\partial y} + \varphi\right)n_2\right] d\Gamma \tag{33}$$

$$+\int_\Omega \tilde{\psi}\left(\frac{\partial w}{\partial x}+\psi\right)+\tilde{\varphi}\left(\frac{\partial w}{\partial y}+\varphi\right)$$
$$-\tilde{w}\left[\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}+\psi\right)+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}+\varphi\right)\right]dxdy\,,$$

where $n=(n_1,n_2)$ denote the outward normal vector to Γ.

Inserting this into (31) we can collect all the terms containing $\tilde{\psi}$, $\tilde{\varphi}$ and $\tilde{w}$ and by applying the fundamental lemma of calculus of variations, we can formulate (31) as the boundary value problem (36).

4. The Mindlin-Timoshenko Plate Model

Introducing again the notation $Q=\Omega\times]0,T[$, $\Sigma_0=\Gamma_0\times]0,T[$ and $\Sigma_1=\Gamma_1\times]0,T[$ we find that

$$\begin{cases}\frac{\rho h^3}{12}\psi''-D(\psi_{xx}+\frac{1-\mu}{2}\psi_{yy}+\frac{1+\mu}{2}\varphi_{xy})+K(\psi+w_x) & =0 \quad \text{in} \quad Q,\\ \frac{\rho h^3}{12}\varphi''-D(\varphi_{yy}+\frac{1-\mu}{2}\varphi_{xx}+\frac{1+\mu}{2}\psi_{xy})+K(\varphi+w_y) & =0 \quad \text{in} \quad Q,\\ \rho h w''-K\left[(\psi+w_x)_x+(\varphi+w_y)_y\right] & =0 \quad \text{in} \quad Q,\end{cases}\tag{34}$$

with the boundary conditions:

$$\begin{cases}u=(\psi,\varphi,w) & =0 \quad \text{on} \quad \Sigma_1,\\ D(n_1\psi_x+\mu n_1\varphi_y+\frac{1-\mu}{2}n_2(\psi_y+\varphi_x)) & =\kappa_1 \quad \text{on} \quad \Sigma_0,\\ D(n_2\varphi_y+\mu n_2\psi_x+\frac{1-\mu}{2}n_1(\psi_y+\varphi_x)) & =\kappa_2 \quad \text{on} \quad \Sigma_0,\\ K(\partial_n w+n_1\psi+n_2\varphi) & =\kappa_3 \quad \text{on} \quad \Sigma_0,\end{cases}\tag{35}$$

and with appropriate initial conditions $u(0)=u^0$ and $u'(0)=u^1$ in Ω. In order to keep a simple notation we will write this as

$$\begin{cases}\mathbf{C}u_{tt}=\mathcal{A}u & \text{in} \quad Q,\\ \mathcal{B}u=\kappa & \text{on} \quad \Sigma_0,\\ u=0 & \text{on} \quad \Sigma_1,\\ (u(0),u_t(0))=(u^0,u^1) & \text{in} \quad \Omega.\end{cases}\tag{36}$$

Here $\partial_n=n_1\frac{\partial}{\partial x}+n_2\frac{\partial}{\partial y}$ and we have written ψ_x instead of $\frac{\partial\psi}{\partial x}$, φ_{xy} for $\frac{\partial^2\varphi}{\partial x\partial y}$, etc., for brevity. Also note that we only apply κ to the part Γ_0 of the boundary.

We usually refer to (36) as the control system since it is the system we want to control by acting on the boundary, i.e. the taskis to find a control κ such that we can steer any initial data to any final state in some suitable function space. The next task is therefore to derive appropriate well-posedness results for (36) and then apply modern control theory to the system. This program is persued in [6], applying the modern formulation of HUM (see [7], [5]).

5. The Variational Formulation and Green's Formula

We will now state a Green's formula associated to system (36), this is *the* fundamental formula which we will use repeatedly throughout the rest of this paper.

We consider (26) and (27) and insert these into (25) and after some rearranging we find

$$\begin{aligned} a(u,v) &= a_0(u,v) + K a_1(u,v) \\ &= -\int_\Omega \tilde{\psi}\left[D\left(\frac{\partial^2\psi}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2\psi}{\partial y^2} + \frac{1+\mu}{2}\frac{\partial^2\varphi}{\partial x\partial y}\right) - K\left(\psi + \frac{\partial w}{\partial x}\right)\right] \\ &\quad + \tilde{\varphi}\left[D\left(\frac{\partial^2\varphi}{\partial y^2} + \frac{1-\mu}{2}\frac{\partial^2\varphi}{\partial x^2} + \frac{1+\mu}{2}\frac{\partial^2\psi}{\partial x\partial y}\right) - K\left(\varphi + \frac{\partial w}{\partial y}\right)\right] \\ &\quad + \tilde{w}K\left(\frac{\partial}{\partial x}\left(\psi + \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varphi + \frac{\partial w}{\partial y}\right)\right) dxdy \\ &\quad + \int_\Gamma \tilde{\psi} D\left[\left(\frac{\partial\psi}{\partial x} + \mu\frac{\partial\varphi}{\partial y}\right)n_1 + \frac{1-\mu}{2}\left(\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial x}\right)n_2\right] \\ &\quad + \tilde{\varphi} D\left[\left(\frac{\partial\varphi}{\partial y} + \mu\frac{\partial\psi}{\partial x}\right)n_2 + \frac{1-\mu}{2}\left(\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial x}\right)n_1\right] \\ &\quad + \tilde{w}K\left[\frac{\partial w}{\partial n} + n_1\psi + n_2\varphi\right] d\Gamma. \end{aligned}$$

Here we recognize $\mathcal{A}u$ and $\mathcal{B}u$ and we get the (halfways) Green's formula:

$$-\int_\Omega \mathcal{A}uv\,dxdy = a(u,v) - \int_{\Gamma_0} \mathcal{B}uv\,d\Gamma\,, \tag{37}$$

where we have assumed $v = 0$ on Γ_1. The complementary boundary operator is then

$$\mathcal{C}v = v|_{\Gamma_0}.$$

5.1. The Variational Form

In order to formulate problem (36) in a variational form we need to construct spaces V and H, such that we have a Gelfand triple

$$V \hookrightarrow H \hookrightarrow V', \tag{38}$$

where as usual $\hookrightarrow$ denotes a continuous, dense injection. We define:

$$H = [L^2(\Omega)]^3, \tag{39}$$

$$H^1_{\Gamma_1}(\Omega) = \left\{ f \in H^1(\Omega) \middle| f = 0 \text{ on } \Gamma_1 \right\} \tag{40}$$

and equip $H^1_{\Gamma_1}$ with the H^1 norm, here $f = 0$ of course has to be taken in sense of traces. Notice that if $\Gamma_1 = \emptyset$ then $H^1_{\Gamma_1}(\Omega) = H^1(\Omega)$ and $\Gamma_1 = \Gamma$ implies $H^1_{\Gamma_1} = H^1_0(\Omega)$. Now we can define V as

$$V = [H^1_{\Gamma_1}(\Omega)]^3, \tag{41}$$

corresponding to the homogeneous Dirichlet boundary condition in (36). Equipping V and H with their respective product topologies we have a continuous, dense injection $V \hookrightarrow H$ and thus the Gelfand triple $V \hookrightarrow H \hookrightarrow V'$. Here we have as usual identified the dual of H with itself. Now it is convenient to introduce the spaces

$$\mathcal{H} = V \times H, \qquad \mathcal{H}' = H \times V'.$$

Now, following the general approach from Pedersen [4], we can formulate (36) in a variational form. From the Green's formula we have

$$\int_\Omega \mathcal{A}uv dxdy + a(u,v) - \int_{\Gamma_0} \mathcal{B}uv d\Gamma = 0$$

using (36) and the expression for c, (29), we get

$$\int_\Omega \mathbf{C}u_{tt}v dxdy + a(u,v) - \int_{\Gamma_0} \mathcal{B}uv d\Gamma = c(u_{tt},v) + a(u,v) - \int_{\Gamma_0} \mathcal{B}uv d\Gamma = 0.$$

From this expression and the spaces defined above we can now write the variational form of the system (36) as

$$\begin{cases} (u,u_t) \in C([0,T];\mathcal{H}), \\ c(u_{tt},v) + a(u,v) - \int_{\Gamma_0} \mathcal{B}uv d\Gamma = 0, \qquad \forall v \in V,\ 0 < t < T, \\ (u(0),u_t(0)) = (u^0,u^1) \in \mathcal{H}. \end{cases} \tag{42}$$

Remark 1. If we consider the equation of motion (31), i.e.

$$\int_0^T \left(c(u_t, v_t) - a(u,v) - \int_{\Gamma_0} \kappa v d\Gamma \right) dt = 0\,,$$

this is equivalent to

$$\frac{d}{dt} c(u_t, v) + a(u,v) + \int_{\Gamma_0} \kappa v d\Gamma = 0 \tag{43}$$

for all

$$v \in \{v \in L^2(0,T,V) | v_t \in L^2(0,T,V), v(0) = v(T) = 0\};$$

here $\frac{d}{dt}$ must be taken in sense of distributions on $]0,T[$.

So from a Lagrange-mechanical point-of-view the formulation of (42) is (43), this is the formulation considered in Lagnese [2], whereas (42) is the formulation used in Lions and Lagnese [3].

6. The Energy

The total energy of the control system (36) is given by

$$E(t) = V_b(t) + T_b(t).$$

Inserting (25) and (29) into (13) and (14), respectively, we get

$$E(t) = \frac{1}{2}[a(u(t), u(t)) + c(u_t(t), u_t(t))]. \tag{44}$$

By differentiation with respect to t we find

$$\begin{aligned} \frac{d}{dt} E(t) &= \frac{1}{2} \frac{d}{dt} [a(u(t), u(t)) + c(u_t(t), u_t(t))] \\ &= [a(u(t), u_t(t)) + c(u_{tt}(t), u_t(t))] \\ &= a(u(t), u_t(t)) + \int_\Omega \mathcal{A} u(t) u_t(t) dx dy \\ &= \int_{\Gamma_0} \mathcal{B} u(t) u_t(t) d\Gamma \\ &= \int_{\Gamma_0} \kappa(t) u_t(t) d\Gamma, \end{aligned}$$

where we have used Green's formula (37).

The energy obviously depends on the control κ, and we see that the energy is preserved in the homogeneous case ($\kappa = 0$) of system (36). Moreover, we see that a natural choice of boundary *feedback* control is $\kappa(t) = -u_t(t)|_{\Gamma_0}$ or, more generally $\kappa(t) = -Fu_t(t)|_{\Gamma_0}$, where F is some positive definite operator on H. This point of view is investigated e.g. in [2]. It is, however, also surprisingly difficult to prove uniform energy decay estimates even in the case above with velocity dependent feedback, where $\frac{d}{dt}E(t) \leq 0$.

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